Chaotic Behavior in Probabilistic Cellular Neural Networks

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Abstract—Experiments conducted in brains by electroencephalograph and magnetoencephalographic techniques reveal widespread coherent oscillations. The oscillations over multiple frequency bands overlap and result in signals with broad spectra. Previous studies showed that various frequencies can be modeled by probabilistic cellular automata with coupled inhibitory and excitatory interactions. In this work we show that coupled oscillator layers can create broad-spectrum chaotic oscillations with power spectral densities over long times segments converging to Brown noise features. Models of cortical neurodynamics provide an interpretation of the observed phenomena.

I. INTRODUCTION

As the connections among biological neurons grow, the neurons start to group themselves in the semi-isolated populations communicating via neural connections [1]. Single neuron activations seem asynchronous, but the activations of neural assemblies exhibit synchrony [30], [27], [11]. Ordinary and partial differential equations used for neural population modeling show complex dynamics, including chaos and chaotic itinerancy [21], [9]. With the brain development, some excitatory or stimulating neurons transform into inhibitory neurons, and as such suppress the activity of neurons they influence. Inhibitory neurons form populations as well. The coupling of inhibitory and excitatory populations leads to the emergence of sustained narrow-band oscillations in the neural tissue [23], [34]. Electroencephalographic (EEG) and magnetoencephalographic (MEG) potentials generated by active brain reveal oscillations overlapping with multiple rhythms [3]. Spectrum analysis of short segments shows peaks in the frequency ranges of the θ (3-7 Hz), α (8-12 Hz), β (13-30), and γ (30-100 Hz) bands. If the duration of segments chosen for analysis increases, the form to which the spectrum converges is a linear decrease in log power with increasing log frequency at a slope near 2 (Brown noise). Any frequency range can be modeled with the probabilistic cellular automata (PCA) composed of the inhibitory (reversely influential) and excitatory sites and local and global (long-range) interactions. PCA are lattice models of spatially extended systems with probabilistic local dynamical rules of evolution [28] and they generalize deterministic cellular automata (DCA) and bootstrap percolations [22], [24], [20], [25], [29], [31]. PCA is difficult to analyze rigorously [6], so the computational simulations provide an alternative tool [13], [15]. Binder’s finite size scaling theory shows that local-global PCA demonstrate behavior consistent with a weak Ising universality class [7], [5], [4]. The role of non-local interactions has been studied in deterministic models as well, such as coupled map lattices (CML) [33]. Stimulation of coupled inhibitory and excitatory populations (oscillator) disturbs the population’s activation amplitudes. Without the stimuli, the population’s activations return to the basal levels. Coupled oscillators of different rhythms create an activation whose spectrum of long segments converges to a desired broad band with the scale-free properties, in the style of brains.

II. COUPLED PCA

A. Node Description

A node, cell, or site is the extreme simplification of a neuron with initial activation 0 or 1. The activation of the i-th node at time t + 1, a_i(t + 1) for t = 0, 1, 2, ... and i = 0, 1, 2, ..., N, depends on 2 factors; influences or links of its neighborhood or the nodes with a direct connection to it at time t, Λ_i, and 0 < ε < 1, where a_i,j(t) is an influence to node i from node j at t with possible values 0 and 1. An example of the influences of neighborhood a_i or Λ_i, could be Λ_i = \{a_i,i-1, a_i,i, a_i,i+1\}.

B. Layer Couplings

A layer is a special 2-dimensional lattice with N^2 nodes. A layer’s topology depends on the nodes’ neighbors. With local topology a node has 5 local neighbors connected by local links; 4 closest in each directions and itself. With non-local topology, some nodes have randomly assigned non-local or remote neighbors connected via remote links. To keep the number of neighbors per node constant, a node loses as many local neighbors as the remote neighbors it gains (Fig. 1 top left). To create a periodic boundary, the layer is folded into a torus. The relative importance of non-local links is defined as n_R = \{# of sites with a non-local link\}/\{# all sites\}×100%. In a single layer, with neighborhoods of 5, nodes influence with 1 when active and 0 when inactive. Update rule is applied simultaneously over all sites at discrete time steps, so equation ?? is reduced to the probabilistic majority rule, according to which the i-th site next value a_i(t + 1) is equal to the majority of its neighbors with probability 1 - ε, and to the minority with probability ε. The layers are coupled if the nodes from one layer neighbor or have cross links to the nodes of another [18], [17]. The higher the node’s number neighboring the other layer’s nodes, the higher the layer’s coupling strength. One of the layers is inhibitory or inversely influential. A node from the inhibitory layer influences the other layer’s node in a reversed manner, influencing with 1 when inactive, and with 0 when active. The other layer’s nodes are excitatory, influence with 1 when active and with 0 when inactive (Fig. 1 top right). A constant size of neighborhood is maintained, so the node loses a
connection (its self-connection) when it is selected to receive connection from the other layer. General parameter describing the layers’s coupling strength $n_L = \{\# of sites with a cross-layer link\}/\{\# all sites\} \times 100\%$. More precisely, $n_{LEI}$ defines amount of layer connections form inhibitory layer to the excitatory. Similarly, $n_{LEI}$ defines amount of layer connections form excitatory layer to the inhibitory. Since coupled excitatory-inhibitory layers can produce oscillation, they are called oscillators. The oscillators are coupled if the nodes from one oscillator neighbor or have cross links to the nodes of another (Fig. 1 bottom). The higher the number of nodes neighboring the other oscillator’s nodes, the higher the oscillator’s coupling strength. General parameter describing the oscillator’s coupling strength $n_O = \{\# of sites with a cross-oscillator link\}/\{\# all sites\} \times 100\%$. Cross-oscillator link’s influence depends on the type of the layer it projects from. More precisely, $n_{OEI,i,j}$ defines amount of layer connections form inhibitory layer of oscillator $j$ to the excitatory layer of oscillator $i$. Similarly, $n_{OEE,i,j}$ defines amount of layer connections form excitatory layer of oscillator $j$ to the excitatory layer of oscillator $i$.

III. ACTIVATION BEHAVIOR OF SINGLE AND COUPLED LAYERS

A. Single Layer

System’s activation, $d = \sum_i a_i/N$, changes with the $\varepsilon$ change and the change of other parameters. Single layer PCA with various update rules and neighborhoods have been studied extensively using renormalization group techniques [10], [15] and Binder’s method [13]. Mixed local and non-local or global neighborhoods show critical behavior and follow finite-size scaling laws. Accordingly, magnetization $d^* \propto (\varepsilon_C - \varepsilon)^{\beta}$, when $\varepsilon \to \varepsilon_C$. Here $d^* = |d - 0.5|$. Similar scaling laws are valid for susceptibility $\chi$ and correlation length $\xi$. The critical exponents satisfy a hyper-scaling identity, which indicates a behavior consistent with weak Ising universality class [4], [5]. Example of the size invariance of quantity $U^*$ is illustrated on Fig. 2 (second from left) in the case of a purely local system. Quantity $U^*$ is related to the kurtosis and it is defined as follows:

$$U^* = \frac{\langle (d - \langle d \rangle)^4 \rangle}{(\langle (d^* - \langle d^* \rangle)^2 \rangle)^2}.$$  

(1)

At the critical point $\varepsilon_C$, $U^*$ is the same for all layer sizes of the same topology [15]. This is the point where the activation densities randomly hover around 0.5 and activation density distributions are uni-modal. For $\varepsilon < \varepsilon_C$, the activations are either mostly active or mostly inactive and the activation distributions are bimodal, as illustrated on the gray shaded histogram on the most left panel of Fig. 2.

B. Oscillator

The behavior of 2-layer perturbed PCA produce damped oscillation (Fig. 3), which is a basic building block of the dynamics of neural populations [30]. After an impulse stimulus, the excitatory cells reach a peak activation. Then the excitatory cells excite inhibitory cells, which reach a peak excitation a quarter of cycle after the excitatory cells peak activity. At this time, the excitatory cells are already inhibited to their basal, resting level. They reach maximum inhibition as the inhibitory cells return to their basal level. During
In this phase, the inhibitory cells fail to receive the excitation from the excitatory cells, so they undergo inhibition. When the excitatory cells are released from inhibition, they again respond to background activity, and start another cycle. Inhibitory connections can generate narrow-band oscillations. There are two critical points \( \varepsilon_O \) and \( \varepsilon_C \). \( \varepsilon_O \) marks the onset of prominent narrow-band oscillations, and \( \varepsilon_C \) describes the transition point where narrow-band oscillations disappear. To achieve the statistical accuracy, experiments are performed on lattice sizes between \( 64 \times 64 \) and \( 128 \times 128 \) for various \( \varepsilon, n_R \) and \( n_L \), and at least for 1 million steps or until \( \langle |0.5 - d| \rangle < 0.001 \). Figure 4 illustrates the onset of prominent oscillations in systems with \( n_R = 25\% \), while \( n_L = 3.125\% \) and \( 12.5\% \). \( n_{LEI} = n_{LIE} \) and \( n_R \) of the layers are the same. Figures show behavior of one layer, layer 0. The other layer’s behavior is the same. At \( \varepsilon = 0.1425 \) and \( n_L = 12.5\% \) (dotted line) PCA has a bimodal distribution and produces prominent narrow-band oscillations. At the same time, at \( n_L = 3.125\% \) (shade) PCA has a more complex quadro-modal distribution without narrow-band oscillations [16]. The top right two plots on Fig. 4 show \( U^* \) values for system size \( 80 \times 80 \) (crosses) and \( 112 \times 112 \) (squares). The curves intersect the first time when activation density distributions transform from quadro-modal to bimodal distribution. This point is marked by \( \langle \varepsilon_O \rangle \). For given \( n_R, \varepsilon_O \) decreases as \( n_L \) increases. For \( \varepsilon < \varepsilon_O \), larger coupled layers have greater \( U^* \). At \( \varepsilon_O \), the \( U^* \) values are the same for any system size. Smaller systems have greater \( U^* \), immediately above \( \varepsilon_O \). By increasing \( \varepsilon > \varepsilon_O \) or \( n_L \), the frequency of the oscillations increases and the distance between the peaks of the activation distribution decreases, (Fig. 5). When \( \varepsilon \) approaches \( \varepsilon_C \), the bimodal activation density distributions transit to uni-modal distributions; see Fig. 4 bottom and left, dotted line. This transition resembles critical phase transitions in single PCA layers, as illustrated on the bottom right panels depicting the evolution of \( U^* \) for systems of different sizes. At the critical conditions \( \varepsilon_C \), \( U^* \) is the same independently of the lattice size, given the same topology. Prominent oscillations exist under the condition \( \varepsilon_O < \varepsilon < \varepsilon_C \).

IV. COUPLED OSCILLATORS

A. Modeling Olfaction

In a single layer, a mutual excitation of excitatory nodes provides the sustained aperiodic activity. The activity of an excitatory layer is self-stabilized by a non-zero point attractor, giving rise to a field of nearly white noise. In the coupled layers with appropriate topology and \( \varepsilon \), inhibitory layer contributes negative feedback, leading to narrow-band oscillations. Coupling the oscillators of different frequencies creates a system with activation whose spectrum of long segments converges to a broad band noise with scale-free properties. There are many possible oscillator couplings. One inspiration comes from Freeman’s K-sets, which model chaotic behavior and are inspired by biological system [2]. K-sets are derived primarily from observations of the olfactory system. K-set building blocks are subsets K0, K1, K2, K3.
Fig. 4. Top: Onset of narrow-band at $\varepsilon_O$ oscillations for $n_R = 25\%$; activation distribution (left) and $U^*$ (right). With greater $n_L$ comes lower $\varepsilon_O$. Bottom: Onset of critical transitions from bimodal to uni-modal states, $n_L = 25\%$; activation distribution (left) and $U^*$ (right). With greater $n_L$ comes lower $\varepsilon_O$.

Fig. 5. Top: By increasing $\varepsilon > \varepsilon_O$ the frequency of oscillations increases; activation distribution (left) and examples of activations and power spectral density estimates on window of 10,000 steps (right) and 1 million step experiment. Bottom: By increasing $n_L$ the frequency of oscillations increases; activation distribution (left) and examples of activations and power spectral density estimates on window of 10,000 steps (right) and 1 million step experiment.
and so on. Higher numbered \( K \) subset is built from lower numbered \( K \) sets. The \( K_0 \)-set integrates a weighted sum of inputs via 2nd order linear differential equation and then applies an asymmetric sigmoid function \( G \). The inputs and outputs of the \( K_0 \) set represent the average firing rate \( (f_{\text{in}}, f_{\text{out}}) \) of a large collection of (about \( 10^5 \)) neurons in the cortex. By coupling a number of excitatory or inhibitory \( K_0 \) sets, \( K_{1E} \) or \( K_{1I} \) are formed. The interactions of \( K_{1E} \) and \( K_{1I} \) layers give the \( K_2 \) set Coupling several \( K_2 \) sets with excitatory, inhibitory, and feedback loops gives the \( K_3 \) set. The \( K_3 \) model is a delay-coupled set of nonlinear oscillators. Harter and Kozma demonstrate the production of deterministic chaos by the \( K_3 \) sets [35]. They used the power spectral density (power spectrum) as a measure of the power of varying frequencies of EEG recording of a biological brain. The typical power spectrum of a rat EEG shows a central peak in the 30-40 Hz range, and a \( 1/f^\alpha \) form of the slope. The measured slope of the power spectrum varies around \( \alpha = 2.0 \). \( 1/f^\alpha \) type power spectra are abundant in nature and are characteristic of critical states, between order and randomness, at which chaotic processes operate. The atypical part of the experimental EEG spectra is the central peak, indicating stronger oscillatory behavior in the \( \gamma \) frequencies and is associated with cognitive processes. PCA models have ability to replicate these types of dynamics, (Fig. 7).

B. Multiple Oscillators

As an example, we could couple 8 oscillators with 2 layers each, one excitatory and one inhibitory. For each layer \( n_R = 25\% \), \( n_L = 25\% \), and \( n_O = 4.6875\% \). In other words, topology is homogenous. Each layer has 15 sets of connections; one excitatory and one inhibitory set from 7 other oscillators. To create different oscillations, the \( \varepsilon \) values are 0.0753125, 0.0859375, 0.0965625, 0.1071875, 0.1178125, 0.1284375, 0.1390625, and 0.1496875 for oscillators 0 through 7 respectively. When the oscillators are isolated without the oscillatory connections, each oscillator has its basal frequency, (Fig. 8). When the oscillators are not too strongly coupled, they influence each other, but not enough to give up their own frequency. Almost all layers show a \( 1/f^{-2} \) form of the slope, which could be indication of chaotic behavior.

V. Conclusion

Rigorous analysis of PCA is difficult [6], so the clues of PCA behavior are given by computational simulations. The evolving excitatory nodes in a single layer show aperiodic activity, which is self-stabilized by a non-zero point attractor. This activity gives rise to a field of nearly white noise. Coupled inhibitory and excitatory layers with appropriate topology and \( \varepsilon \) show narrow-band activity oscillations. The oscillators can model any frequency range. Through \( U^* \) and activation distribution critical points \( \varepsilon_O \) and \( \varepsilon_C \) are nicely marked. \( \varepsilon_O \) marks the onset and \( \varepsilon_C \) marks the disappearance of prominent narrow-band oscillations. Coupled oscillators
of different frequencies show activations whose spectra converge to a broad band noise with scale-free properties. PCA models have ability to replicate the dynamics of K3 sets, which demonstrated deterministic chaos [35]. Case studies, in which the oscillators are not too strongly coupled so they influence each other, but not overly to lose they own frequencies, show a $1/f^{-2}$ form of the slope. This could be indication of chaotic behavior. The intrinsic chaotic nature of the dynamics of coupled PCA oscillators will be evaluated in future studies.

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