The Invariant Subspace Problem: General Operator Theory vs. Concrete Operator Theory?

Problems and Recent Methods in Operator Theory in Memory of Prof. James Jamison

Memphis, October 2015
Introduction
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\[ T : \mathcal{H} \rightarrow \mathcal{H}, \text{linear and bounded} \]
\[ T(M) \subset (M), \text{closed subspace} \]
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- **Remarks**

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1. Finite dimensional complex Hilbert spaces.
Example: $\mathbb{R}^2$

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

respect to the canonical bases $\{e_1, e_2\}$. 
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$T$ has no non-trivial invariant subspaces in $\mathbb{R}^2$
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- $\ell^2 = \{\{a_n\}_{n \geq 1} \subset \mathbb{C} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$

$\{e_n\}_{n \geq 1}$ canonical bases in $\ell^2$
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\( \{e_n\}_{n \geq 1} \) canonical bases in \( \ell^2 \)

\[ Se_n = e_{n+1} \quad n \geq 1 \]
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Characterization of the invariant subspaces of \( S \)?
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Characterization of the invariant subspaces of $S$?

$\ker(S - \lambda I) = \{0\}$ for any $\lambda \in \mathbb{C}$.
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Characterization of the invariant subspaces of \( S \)?

\( \ker(S - \lambda I) = \{0\} \) for any \( \lambda \in \mathbb{C} \). That is, \( \sigma_p(S) = \emptyset \).
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Characterization of the invariant subspaces of \( S \)?

Classical Beurling Theory:
Inner-Outer Factorization of the functions in the Hardy space.
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• Classes of operators with known invariant subspaces:
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  ★ Normal operators (Spectral Theorem)
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  - ★ Normal operators (Spectral Theorem)
  - ★ Compact Operators
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★ 1951, J. von Newman, (Hilbert space case)
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  ★ 1954, Aronszajn and Smith (general case)
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    - 1966, Bernstein and Robinson, (Hilbert space case)
    - 1967, Halmos
    - 1960’s, Gillespie, Hsu, Kitano...
In the Banach space setting

In 1975, P. Enflo showed in the *Semiaire Maurey-Schwartz* at the École Polytechnique in París:
In the Banach space setting

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*There exists a separable Banach space $\mathcal{B}$ and a linear, bounded operator $T$ acting on $\mathcal{B}$, injective and with dense range, without no non-trivial closed invariant subspaces.*
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There exists a separable Banach space $B$ and a linear, bounded operator $T$ acting on $B$, **injective and with dense range**, without no non-trivial closed invariant subspaces.

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There exists a separable Banach space $B$ and a linear, bounded operator $T$ acting on $B$, *injective and with dense range*, without non-trivial closed invariant subspaces.


• 1985, C. Read, Construction of a linear bounded operator on $\ell^1$ without non-trivial closed invariant subspaces.
The big open question
The big open question

Does every linear bounded operator $T$ acting on a separable, reflexive complex Banach space $\mathcal{B}$ (or a Hilbert space $\mathcal{H}$) have a non-trivial closed invariant subspace?
Classes of operators with known invariant subspaces

- 1973, Lomonosov
Classes of operators with known invariant subspaces

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**Theorem (Lomonosov)** Let $T$ be a linear bounded operator on $\mathcal{H}$, $T \neq \mathbb{C}Id$. If $T$ commutes with a non-null compact operator, then $T$ has a non-trivial closed invariant subspace.
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**Theorem (Lomonosov)** Let $T$ be a linear bounded operator on $\mathcal{H}$, $T \neq \mathbb{C}Id$. If $T$ commutes with a non-null compact operator, then $T$ has a non-trivial closed invariant subspace. Moreover, $T$ has a non-trivial closed hyperinvariant subspace.

**Theorem (Lomonosov)** Any linear bounded operator $T$, not a multiple of the identity, has a nontrivial invariant closed subspace if it commutes with a non-scalar operator that commutes with a nonzero compact operator.
Classes of operators with known invariant subspaces
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- Does every operator satisfy “Lomonosov Hypotheses”? 
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- 1980, Hadwin; Nordgren; Radjavi y Rosenthal
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Construction of a "quasi-analytic" shift $S$ on a weighted $\ell^2$ space which has the following property: if $K$ is a compact operator which commutes with a nonzero, non scalar operator in the commutant of $S$, then $K = 0$. 
A “Concrete Operator Theory” approach

- Universal Operators (in the sense of G. C. Rota)
A "Concrete Operator Theory" approach

- **Universal Operators** (in the sense of G. C. Rota)

  A linear bounded operator $U$ in a Hilbert space $\mathcal{H}$ is **universal** if for any linear bounded operator $T$ in $\mathcal{H}$, there exists $\lambda \in \mathbb{C}$ and $M \in \text{Lat}(U)$ such that $\lambda T$ is similar to $U|_M$. 
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- **Example.** Adjoint of a unilateral shift of infinite multiplicity.
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- **Example.** Adjoint of a unilateral shift of infinite multiplicity. It may be regarded as $S^*$ in $(\ell^2(\mathcal{H}))$ defined by

  $$S^*((h_0, h_1, h_2, \cdots)) = (h_1, h_2, \cdots)$$

for $(h_0, h_1, h_2, \cdots) \in \ell^2(\mathcal{H})$. 
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• **Example.** Let $a > 0$ and $T_a : L^2(0, \infty) \rightarrow L^2(0, \infty)$ defined by

$$T_afa(t) = f(t + a), \quad \text{for } t > 0.$$

$T_a$ is universal.
A “Concrete Operator Theory” approach: universal operators

- **Proposition.** Let $\mathcal{H}$ be a Hilbert space and $U$ a linear bounded operator. Suppose that $U$ is a universal operator. The following conditions are equivalent:
A “Concrete Operator Theory” approach: universal operators

• **Proposition.** Let $\mathcal{H}$ be a Hilbert space and $U$ a linear bounded operator. Suppose that $U$ is a universal operator. The following conditions are equivalent:

1. Every linear bounded operator $T$ on $\mathcal{H}$ has a non-trivial closed invariant subspace.
A “Concrete Operator Theory” approach: universal operators

- **Proposition.** Let $H$ be a Hilbert space and $U$ a linear bounded operator. Suppose that $U$ is a universal operator. The following conditions are equivalent:

  1. Every linear bounded operator $T$ on $H$ has a non-trivial closed invariant subspace.

  2. Every closed invariant subspace $M$ of $U$ of dimension greater than 1 contains a proper closed and invariant subspace.
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- Proposition. Let $\mathcal{H}$ be a Hilbert space and $U$ a linear bounded operator. Suppose that $U$ is a universal operator. The following conditions are equivalent:

1. Every linear bounded operator $T$ on $\mathcal{H}$ has a non-trivial closed invariant subspace.

2. Every closed invariant subspace $M$ of $U$ of dimension greater than 1 contains a proper closed and invariant subspace (i.e. the minimal non-trivial closed and invariant subspaces for $U$ are one-dimensional).
Providing universal operators

• **Universal Operators** (in the sense of G. C. Rota)

A linear bounded operator $U$ in a Hilbert space $\mathcal{H}$ is **universal** if for any linear bounded operator $T$ in $\mathcal{H}$, there exists $\lambda \in \mathbb{C}$ and $M \in \text{Lat}(U)$ such that $\lambda T$ is similar to $U|_M$, i.e., $\lambda T = J^{-1}UJ$ where $J : \mathcal{H} \to M$ is a linear isomorphism.

1. Ker($U$) is infinite dimensional,
2. $U$ is surjective.
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Let $U$ be a linear bounded operator on a Hilbert space. Assume that

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Then $U$ is universal.
Idea of the Proof

Write $\mathcal{K} \rightleftharpoons \text{Ker } U$

1. $UV = \text{Id}$,
2. $UW = 0$,
3. $\ker W = \{0\}$,
4. $\text{Im } W = \mathcal{K}$ and $\text{Im } V = \mathcal{K}^\perp$.

- $\mathcal{M} = \text{Im } J$ is a closed subspace of $U$.
- $J$ is an isomorphism onto $\mathcal{M}$. 
Idea of the Proof

Write $\mathcal{K} = \text{Ker } U$

**Step 1:** Construct $V, W \in L(\mathcal{H})$ such that

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**Step 2:** Prove that $U$ is universal.

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Let $T$ be a linear bounded operator on $\mathcal{H}$.

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**Step 1:** Construct $V, W \in \mathcal{L}(\mathcal{H})$ such that

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**Step 2:** Prove that $U$ is universal.

Let $T$ be a linear bounded operator on $\mathcal{H}$.

Let $\lambda \neq 0$ such that $|\lambda| \|T\| \|U\| < 1$ and define

$$J = \sum_{k=0}^{\infty} \lambda^k V^k W T^k.$$

- $\mathcal{M} = \text{Im } J$ is a closed subspace of $U$.
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Idea of the Proof

Write $\mathcal{K} = \text{Ker } \mathcal{U}$

**Step 1:** Construct $V, W \in \mathcal{L}(\mathcal{H})$ such that

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**Step 2:** Prove that $\mathcal{U}$ is universal.

Let $T$ be a linear bounded operator on $\mathcal{H}.$

Let $\lambda \neq 0$ such that $|\lambda| \|T\| \|\mathcal{U}\| < 1$ and define

$$J = \sum_{k=0}^{\infty} \lambda^k V^k W T^k.$$  

$J$ satisfies $J = W + \lambda V J T$ and therefore, $UJ = \lambda JT.$

- $\mathcal{M} = \text{Im } J$ is a closed subspace of $\mathcal{U}.$
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Idea of the Proof

Write \( \mathcal{K} = \ker U \)

**Step 1:** Construct \( V, W \in \mathcal{L}(\mathcal{H}) \) such that

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2. \( UW = 0 \),
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**Step 2:** Prove that \( U \) is universal.

Let \( T \) be a linear bounded operator on \( \mathcal{H} \).

Let \( \lambda \neq 0 \) such that \( |\lambda| \|T\| \|U\| < 1 \) and define

\[
J = \sum_{k=0}^{\infty} \lambda^k V^k W T^k.
\]

\( J \) satisfies \( J = W + \lambda VJT \) and therefore, \( UJ = \lambda JT \). In addition,

- \( \mathcal{M} = \text{Im} J \) is a closed subspace of \( U \).
- \( J \) is an isomorphism onto \( \mathcal{M} \).
Examples of universal operators

- **Theorem** (1987, Nordgren, Rosenthal y Wintrobe)
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Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$. For every $\lambda$ in the interior of the spectrum of $C_\varphi$, $C_\varphi - \lambda I$ is universal in $H^2$. 
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Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$. For every $\lambda$ in the interior of the spectrum of $C_\varphi$, $C_\varphi - \lambda I$ is universal in $\mathcal{H}^2$.

- **Disc automorphisms**

$$
\varphi(z) = e^{i\theta} \frac{p - z}{1 - \overline{p}z} \quad (z \in \mathbb{D}).
$$

where $p \in \mathbb{D}$ and $-\pi < \theta \leq \pi$. 
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- **Parabolic.** $\varphi$ has just one fixed point $\alpha \in \partial \mathbb{D}$ ($\iff |p| = \cos(\theta/2)$)
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Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$. For every $\lambda$ in the interior of the spectrum of $C\varphi$, $C\varphi - \lambda I$ is universal in $\mathcal{H}^2$.

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  * **Parabolic.** $\varphi$ has just one fixed point $\alpha \in \partial\mathbb{D}$ ($\iff |p| = \cos(\theta/2)$)
  * **Hyperbolic.** $\varphi$ has two fixed points $\alpha$ and $\beta$, such that $\alpha, \beta \in \partial\mathbb{D}$ ($\iff |p| > \cos(\theta/2)$)
  * **Elliptic.**
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Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \). For every \( \lambda \) in the interior of the spectrum of \( C_\varphi \), \( C_\varphi - \lambda I \) is universal in \( \mathcal{H}^2 \).

- **Disc automorphisms**

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\varphi(z) = e^{i\theta} \frac{p - z}{1 - \overline{p}z} \quad (z \in \mathbb{D}).
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where \( p \in \mathbb{D} \) and \(-\pi < \theta \leq \pi\).

- **Parabolic.** \( \varphi \) has just one fixed point \( \alpha \in \partial \mathbb{D} \) (\( \Leftrightarrow |p| = \cos(\theta/2) \))

- **Hyperbolic.** \( \varphi \) has two fixed points \( \alpha \) and \( \beta \), such that \( \alpha, \beta \in \partial \mathbb{D} \) (\( \Leftrightarrow |p| > \cos(\theta/2) \))

- **Elliptic.** \( \varphi \) has two fixed points \( \alpha \) and \( \beta \), with \( \alpha \in \mathbb{D} \) (\( \Leftrightarrow |p| < \cos(\theta/2) \))
Examples of universal operators

- **Theorem** (1987, Nordgren, Rosenthal y Wintrobe)

Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$. For every $\lambda$ in the interior of the spectrum of $C_\varphi$, $C_\varphi - \lambda I$ is universal in $\mathcal{H}^2$.

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Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \).

We may assume that \( \varphi \) fixes 1 and \(-1\). Then,

\[
\varphi(z) = \frac{z + r}{1 + rz}, \quad 0 < r < 1.
\]
Examples of universal operators

Every linear bounded operator $T$ has a closed non-trivial invariant subspace
Examples of universal operators

Every linear bounded operator $T$ has a closed non-trivial invariant subspace

$$\uparrow$$

for any $f \in \mathcal{H}^2$, not an eigenfunction of $C_\varphi$, there exists $g \in \overline{\text{span}\{C_\varphi^n f : n \geq 0\}}$ such that $g \neq 0$ and

$$\overline{\text{span}\{C_\varphi^n g : n \geq 0\}} \neq \overline{\text{span}\{C_\varphi^n f : n \geq 0\}}.$$
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\[ \text{span}\{C_\varphi^n g : n \geq 0\} \neq \text{span}\{C_\varphi^n f : n \geq 0\}. \]

\[ \forall \]

the **minimal** non-trivial closed invariant subspaces for $C_\varphi$ are one-dimensional
Which conditions on $f$ ensure that $K_f := \overline{\text{span}}\{C^n\phi f : n \geq 0\}$ is (or not) minimal?
**Minimal invariant subspaces**

**Theorem** (Matache, 1993) Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \) and 1 one of the fixed points of \( \varphi \). Let \( f \in \mathcal{H}^2 \) a non-constant function such that \( f \) extends continuously to 1 and \( f(1) \neq 0 \). Then \( K_f \) is not minimal.

**Theorem** (Mortini, 1995) Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \) and 1 the attractive fixed point of \( \varphi \). Let \( f \in \mathcal{H}^\infty \) a non-constant function such that there exists the radial limit at 1 and \( f(1) \neq 0 \). Then \( K_f \) is not minimal.

**Theorem** (Matache, 1998) Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \) and 1 one of the fixed points of \( \varphi \). Let \( f \in \mathcal{H}^2 \) a non-constant function such that there exists the radial limit of \( f \) at 1 and \( f(1) \neq 0 \). Assume that \( f \) is essentially bounded in an arc \( \gamma \subset \partial \mathbb{D} \) so that \( 1 \in \gamma \). Then \( K_f \) is not minimal.
Minimal invariant subspaces

1. \( \lim_{z \to -1} |f(z)| < \infty \),

2. \( |f(z)| \leq C|z - 1|^\varepsilon \) for some constant \( C \), and \( \varepsilon > 0 \) in a neighborhood of 1.
Minimal invariant subspaces

**Theorem** (Chkliar, 1997) Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \) with fixed points 1 and \(-1\). Assume 1 is the attractive fixed point of \( \varphi \). Suppose that \( f \in \mathcal{H}^2 \) such that

1. \( \lim_{z \to -1} |f(z)| < \infty \),

2. \( |f(z)| \leq C|z - 1|^\varepsilon \) for some constant \( C \), and \( \varepsilon > 0 \) in a neighborhood of 1.

Then the point spectrum of \( C\varphi \) acting on \( \text{span}\{C^n f : n \in \mathbb{Z}\} \mathcal{H}^2 \) contains the annulus

\[
\{ z \in \mathbb{C} : |\varphi'(1)|^{\min\{\varepsilon, \frac{1}{2}\}} < |z| < 1 \},
\]

except, possibly, a discrete subset.
Minimal invariant subspaces

- **Theorem** (2011, Shapiro) Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$ with fixed points 1 and $-1$.

1. If $f \in \sqrt{(z - 1)(z + 1)} \mathcal{H}^2 \setminus \{0\}$, then the point spectrum of $C_\varphi$ acting on $\overline{\text{span}\{C^n_\varphi f : n \in \mathbb{Z}\}} \mathcal{H}^2$ intersects the unit circle in a set of positive measure.

2. If $f \in \sqrt{(z - 1)(z + 1)} \mathcal{H}^p \setminus \{0\}$ for some $p > 2$, then the point spectrum of $C_\varphi$ acting on $\overline{\text{span}\{C^n_\varphi f : n \in \mathbb{Z}\}} \mathcal{H}^2$ contains an open annulus centered at the origin.
Minimal invariant subspaces

**Theorem.** (2011, GG-Gorkin) Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$ fixing 1 and $-1$. Assume that 1 is the attractive fixed point. Let $f$ be a nonzero function in $\mathcal{H}^2$ that is continuous at 1 and $-1$ and such that $f(1) = f(-1) = 0$. Then there exists $g \in \mathcal{H}^2$ satisfying the following conditions:

1. $g \in K_f := \text{span}\{C^nf : n \geq 0\}^{\mathcal{H}^2}$.
2. There exists a subsequence $\{\varphi_{n_k}\}$ such that $g \circ \varphi_{-n_k}$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.
3. $g$ has no radial limit at $-1$. 
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Furthermore, if $f$ belongs to the disc algebra $\mathcal{A}(\mathbb{D})$, then $g \in \mathcal{H}^\infty$ and, consequently, if $K_f$ is minimal, then $K_g = K_f$. 
Question

Eigenfuntions of $C_\varphi$?
Question

Eigenfunctions of $C_\varphi$?

- 2012, GG, Gorkin and Suárez, Constructive characterization of eigenfunctions of $C_\varphi$ in the Hardy spaces $\mathcal{H}^p$
Universal operators vs. Lomonosov Theorem
Universal operators vs. Lomonosov Theorem

- **Naive Question**: Does there exist a universal operator which commutes with a non-null compact operator *in a non-trivial way*?
**Universal operators**

Suppose $S$ is a multiplication operator on the Hardy space $\mathcal{H}^2$ whose symbol is a singular inner function or infinite Blaschke product.

1. $S$ is an isometric operator.
2. $S^*$ has infinite dimensional kernel and maps $\mathcal{H}^2$ onto $\mathcal{H}^2$. 
Universal operators

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Remarks

- $S^*$ is universal.
- Using the Wold Decomposition Theorem, such an operator can be represented as a block matrix on $\mathcal{H} = \bigoplus_{k=1}^{\infty} S^k W$, where $W = \mathcal{H}^2 \ominus S\mathcal{H}^2$. Such a matrix is an upper triangular and has the identity on the super-diagonal:
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$$
S^* \sim \begin{pmatrix}
0 & I & 0 & 0 & \cdots \\
0 & 0 & I & 0 & \cdots \\
0 & 0 & 0 & I & \cdots \\
& & & & \ddots
\end{pmatrix}
$$
Universal operators

An easy computation shows that every operator that commutes with $S^*$ has the form

- This is an upper triangular block matrix whose entries on each diagonal are the same operator on the infinite dimensional Hilbert space $W$.
- Every block in such a matrix occurs infinitely often.
- So, the only compact operator that commutes with the universal operator $S^*$ is 0,
Universal operators

An easy computation shows that every operator that commutes with \( S^* \) has the form

\[
A \sim \begin{pmatrix}
A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\
0 & A_0 & A_{-1} & A_{-2} & \cdots \\
0 & 0 & A_0 & A_{-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

an upper triangular block Toeplitz matrix.

Observe that:

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- Every block in such a matrix occurs infinitely often.
- So, the only compact operator that commutes with the universal operator $S^*$ is 0, not an interesting compact operator!
Universal operators

- **Theorem** (2011, Cowen-GG)

Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \). Then \( C^*_\varphi \) is similar to the Toeplitz operator \( T_\psi \), where \( \psi \) is the covering map of the unit disc onto the interior of the spectrum \( \sigma(C_\varphi) \).
**Universal operators**

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- **Theorem** (1980, Cowen)

  A Toeplitz operator $T_\psi$ in $H^2$, where $\psi \in H^\infty$ is an inner function or a covering map commutes with a compact operator $K$ if and only if $K = 0$. 
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• **Straightforward consequence**

Known universal operators are not commuting with non-null compact operators.
Universal operators
Universal operators

- **Theorem [2013, Cowen-GG]** There exists a universal operator which commutes with an injective, dense range compact operator.
A universal operator which commutes with a compact operator
Compact operators commuting with universal operators

We have seen that some universal operators commute with a compact operator and others do not.
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**Observation:** There are many more compact operators than just one commuting with the universal operator $T^*_\varphi$.

**Definition.** Let $\mathcal{K}_\varphi$ be the set of compact operators that commute with $T^*_\varphi$, that is,

$$\mathcal{K}_\varphi = \{ G \in B(H^2) : G \text{ is compact, and } T^*_\varphi G = GT^*_\varphi \}$$
Compact operators commuting with universal operators

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**Remark.** $\mathcal{K}_\varphi \neq (0)$. 
If $F$ is a bounded operator on $H^2$, we will write $\{F\}'$ for the commutant of $F$, the set of operators that commute with $F$, that is,

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For any operator $F$, the commutant $\{F\}'$ is a norm-closed subalgebra of $\mathcal{B}(H^2)$. 
Theorem [2015, Cowen, GG] The set $\mathcal{K}_\phi$ is a closed subalgebra of $\{T_\phi^*\}'$ that is a two-sided ideal in $\{T_\phi^*\}'$. In particular, if $G$ is a compact operator in $\mathcal{K}_\phi$ and $g$ and $h$ are bounded analytic functions on the disk, then $T_g^*G$, $GT_h^*$, and $T_g^*GT_h^*$ are all in $\mathcal{K}_\phi$. Moreover, every operator in $\mathcal{K}_\phi$ is quasi-nilpotent.
Let $A$ be a linear bounded operator on a Hilbert space and $T$ a universal operator which commutes with a compact operator $W$. 
Consequences, Further Observations, and a Question

Let $A$ be a linear bounded operator on a Hilbert space and $T$ a universal operator which commutes with a compact operator $W$.

- WLOG $A$ is the restriction of $T$ to $M$. 
Consequences, Further Observations, and a Question

Let $A$ be a linear bounded operator on a Hilbert space and $T$ a universal operator which commutes with a compact operator $W$.

- WLOG $A$ IS the restriction of $T$ to $M$.
- WLOG $M \neq \mathcal{H}$ because if so, Lomonosov gives hyperinvariant subspace.
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• WLOG $T$ and $A$ are invertible: replace $T$ by $T + (1 + \|T\|)I$
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- $\mathcal{H} = M \oplus M^\perp$ and with respect to this decomposition

\[ T \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad W \sim \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \]

where $A$, $C$ are invertible and $P$, $Q$, $R$, $S$ are compact.
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where $A$, $C$ are invertible and $P$, $Q$, $R$, $S$ are compact.
- NOT $P = 0$ and $R = 0$ because $\text{kernel}(W) = (0)$.
Consequences, Further Observations, and a Question

- From the relation, $TW = WT$, it follows that

$$AP + BR = PA \quad \text{and} \quad CR = RA$$
Consequences, Further Observations, and a Question

- From the relation, $TW = WT$, it follows that

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Observation: Since $A$ is the operator of primary interest, Equation

$$AP + BR = PA$$

is not so interesting if $P = 0$. 
Lemma. If the universal operator $T = T^*$ and the compact operator $W = W^*_{\psi,J}$ have the representations

$$T \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad W \sim \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

respect $\mathcal{H} = M \oplus M^\perp$, then there are a universal operator $\tilde{T}$ and an injective compact operator $\tilde{W}$ with dense range that commute for which $\tilde{P}$ in a replacement of $P$ is not zero, that is, without loss of generality, we may assume $P \neq 0$. 

Consequences, Further Observations, and a Question
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**Theorem [Cowen, GG]** Let the universal operator $T$ and the commuting injective compact operator $W$ with dense range having the representations with $P \neq 0$. Then the following are true:

- Either $R \neq 0$ or $A$ has a nontrivial hyperinvariant subspace.
- Either $\ker(R) = (0)$ or $A$ has a nontrivial invariant subspace.
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Theorem [Cowen, GG] Suppose \( L \) is an invariant subspace for the universal operator \( T^*_\varphi \) and the block matrix

\[
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix}
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represents \( T^*_\varphi \) based on the splitting \( H^2 = M \oplus M^\perp \). Then, the projection of \( L^\perp \) onto \( M \) is an invariant linear manifold for \( A^* \), the adjoint of the restriction of \( T^*_\varphi \) to \( M \).
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Remark.
Theorem [Cowen, GG] Suppose \( L \) is an invariant subspace for the universal operator \( T^*_\varphi \) and the block matrix

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Remark. Any of the linear manifolds provided by this Theorem are proper and invariant but, in principle, they are not necessarily non-dense.
Consequences, Further Observations, and a Question

Question: Is any of those proper $A^*$-invariant linear manifolds non-dense?
Bibliography (basic)


Bibliography (basic)


