Narrow operators

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**Definition**

An function $x$ is called a **sign on a set** $A$ if $x$ takes values in the set $\{-1, 0, 1\}$ and $\text{supp}x = A$.

$x$ is a **sign of mean zero** if $\int_{\Omega} x \, d\mu = 0$.

**Definition (Plichko, Popov, 1990)**

Let $E$ be a Köthe function space and $X$ be a Banach space. An operator $T : E \rightarrow X$ is called **narrow** if for each set $A$ with positive measure, and for each $\varepsilon > 0$ there exists a mean zero sign $x$ on $A$ such that

$$\|Tx\| < \varepsilon \|x\|.$$
Can we adapt the definition of narrowness to avoid using non-continuous functions?
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Is it enough to require the existence of a function $x$ in $E$ with milder restrictions on its distribution than the requirement that $x^2$ is the characteristic function of a subset $A$?
Definition: For any measurable function $x$ and $M > 0$ we define the $M$-truncation $x^M$ of $x$ as

$$x^M(t) = \begin{cases} 
    x(t), & \text{if } |x(t)| \leq M, \\
    M \cdot \text{sign}(x(t)), & \text{if } |x(t)| > M.
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Definition: Let $1 < p \leq 2$. A function $\varphi : (0, +\infty) \rightarrow [0, 1]$ is said to be $p$-gentle if $\varphi$ is decreasing and

$$\lim_{M \to +\infty} M^{2-p} (\varphi(M))^p = 0.$$
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**Definition:** Let $1 < p \leq 2$, $X$ a Banach space. An operator $T \in \mathcal{L}(L^p, X)$ is called gentle-narrow if there exists a $p$-gentle function $\varphi : (0, +\infty) \to [0, 1]$ such that $\forall \varepsilon > 0 \forall M > 0 \forall A \in \Sigma^+$ there exists $x \in L^p$ such that

(i) $x \neq 0$ and $\text{supp } x \subseteq A,$

(ii) $\|x - x^M\| \leq \varphi(M)\|x\|,$

(iii) $\|Tx\| \leq \varepsilon\|x\|.$
For example an operator $T \in \mathcal{L}(L_p, X)$, $1 < p \leq 2$, is gentle-narrow if $\forall A \in \Sigma \forall \varepsilon > 0$ there exists a mean zero gaussian random variable $x \in L_p(A)$ with distribution

$$d_x \overset{\text{def}}{=} \mu \{ x < a \} = \frac{\mu(A)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{a} e^{-\frac{t^2}{2\sigma^2}} dt,$$

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(Use $\varphi(M) = C e^{-\frac{M^2}{2\sigma^2}}$, where $C$ is a constant independent of $M$.)
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Note that every narrow operator is gentle-narrow with

$$\varphi(M) = \begin{cases} 1 - M, & \text{if } 0 \leq M < 1, \\ 0, & \text{if } M \geq 1. \end{cases}$$
Theorem (Mykhaylyuk, Popov, BR, Schechtman, 2012)

Let $1 < p \leq 2$. Then an operator $T \in \mathcal{L}(L_p)$ is narrow if and only if it is gentle-narrow.
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(i) \(x \neq 0\) and \(\text{supp } x \subseteq A\),
(ii) \(\|x - x^M\| \leq \varphi(M)\|x\|\),
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Question: Is it possible to replace conditions (i)-(iii) with conditions which only use functions with full support?
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**Question:** Is it possible to replace conditions (i)-(iii) with conditions which only use functions with full support?

**Question 2:** Is it possible to find an analogous characterization for $p > 2$? (This could be harder.)
Compact operators are narrow.
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The sequence of Rademachers \((r_n)\) on set \(A\) is weakly null, so, for a compact operator \(T\), the set \(\{Tr_n : n \in \mathbb{N}\}\) is relatively compact and thus \(\|Tr_{n_k}\| \longrightarrow 0\).
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Converse does not hold
Not every narrow operator is compact
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Consider the conditional expectation operator

\[ M^F x = \sum_{i \in I} \left( \frac{1}{\mu(A_i)} \int_{A_i} x \, d\mu \right) \cdot \chi_{A_i} \]
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If \( M^\mathcal{F} \) is well defined on \( E \) and bounded then it is narrow (it is easy to construct mean zero signs that are mapped to zero).
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$$M^F x = \sum_{i \in I} \left( \frac{1}{\mu(A_i)} \int_{A_i} x \, d\mu \right) \cdot \chi_{A_i}$$

If $M^F$ is well defined on $E$ and bounded then it is narrow (it is easy to construct mean zero signs that are mapped to zero).

If $I$ is infinite, then $M^F$ is non-compact.

For example, on $L_p$, $M^F$ is a projection onto $\ell_p$, so $M^F$ is an isomorphism on an infinite dimensional subspace of $L_p$. 
Why do we study narrow operators?

Ultimate answer for operators from $L_1[0, 1]$ to $L_1[0, 1]$.

**Theorem (Rosenthal, 1984)**

An operator $T : L_1 \rightarrow L_1$ is narrow if and only if, for each measurable set $A \subseteq [0, 1]$ the restriction $T|_{L_1(A)}$ is not an isomorphic embedding.
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Let me rephrase this:

Theorem (Rosenthal, 1984)

For an operator $T : L_1 \to L_1$ TFAE:

(i) $T$ is non-narrow,

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Proof: (ii) $\implies$ (i) Clear.
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Proof: (ii) $\implies$ (i) Clear. (i) $\implies$ (ii) Homework.
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This question is open. However we do know that Rosenthal’s Theorem does not generalize to spaces $L_p[0, 1]$ when $2 < p < \infty$.

**Example**

Let $2 < p < \infty$ and

$$T = S \circ J,$$

where $J : L_p \to L_2$ is the formal identity embedding, and $S : L_2 \to L_p$ is an isomorphic embedding.

Then $T$ is not narrow, and for all $A \subseteq [0, 1]$, $T|_{L_p(A)}$ is not an isomorphism.
Open Problem

Is Rosenthal’s Theorem true in $L_p$, for $1 < p \leq 2$?
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For all $1 < p < 2$, there exists a constant $K_p \in \mathbb{R}$ so that for every non-narrow operator $T : L_p \rightarrow L_p$, there exists a subspace $Z \subset L_p$ isomorphic to $L_p$, so that $T|_Z$ is an isomorphism on $Z$. 
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Moreover, $Z$ is $K_p$-complemented in $L_p$, $X$ is $K_p$-isomorphic to $L_p$, $T|_Z$ is a $K_p$-isomorphism, and $T(Z)$ is $K_p$-complemented in $L_p$.
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Theorem (Johnson, Maurey, Schechtman, Tzafriri, 1979; noticed by Bourgain in 1981; recent proof of the full strength of the statement in Dosev, Johnson, Schechtman, 2011, see also book by Popov, BR 2013)

*For all $1 < p < 2$, there exists a constant $K_p \in \mathbb{R}$ so that for every non-narrow operator $T : L_p \to L_p$, there exists a subspace $Z \subset L_p$ isomorphic to $L_p$, so that $T|_Z$ is an isomorphism on $Z$.*

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Natural question

Since we can’t resolve the BIG QUESTION that I stated above we consider the following natural question arising from it.
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Problem

Let $E$ be a function space, $X$ a Banach space, and suppose that $T \in \mathcal{L}(E, X)$ is non-narrow. Does this imply that there exists an infinite dimensional nice subspace $Z \subseteq E$ so that $T|_Z$ is an isomorphism onto $T(Z)$?
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What does it mean nice?
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Sometimes we are happy just to get any infinite dimensional subspace.
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Definition of strictly singular operators

Definition

Let $E$ be a function space, $Z$ a subspace of $E$, $X$ a Banach space and $T \in \mathcal{L}(E, X)$. We say that

- $T$ fixes a copy of $Z$ if there exists a subspace $Z_1 \subseteq E$ so that $Z_1$ is isomorphic to $Z$, and $T|_{Z_1}$ is an isomorphism onto $T(Z_1)$,
Definition of strictly singular operators

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- $T$ fixes a copy of $Z$ if there exists a subspace $Z_1 \subseteq E$ so that $Z_1$ is isomorphic to $Z$, and $T|_{Z_1}$ is an isomorphism onto $T(Z_1)$,

- $T$ is $Z$-strictly singular if $T$ does not fix $Z$. 
Definition of strictly singular operators

Let $E$ be a function space, $Z$ a subspace of $E$, $X$ a Banach space and $T \in \mathcal{L}(E, X)$. We say that

- $T$ fixes a copy of $Z$ if there exists a subspace $Z_1 \subseteq E$ so that $Z_1$ is isomorphic to $Z$, and $T|_{Z_1}$ is an isomorphism onto $T(Z_1)$,
- $T$ is $Z$-strictly singular if $T$ does not fix $Z$,
- $T$ is strictly singular if for every infinite dimensional subspace $W$ of $E$, $T|_W$ is not an isomorphism.
Thus our natural question:
Suppose that $T \in \mathcal{L}(E, X)$ is non-narrow. Does this imply that there exists an infinite dimensional nice subspace $Z \subseteq E$ so that $T|_Z$ is an isomorphism onto $T(Z)$?

can be phrased as:
Thus our natural question:
Suppose that $T \in \mathcal{L}(E, X)$ is non-narrow. Does this imply that there exists an infinite dimensional nice subspace $Z \subseteq E$ so that $T|_Z$ is an isomorphism onto $T(Z)$?

can be phrased as:

**Problem**
- Is every strictly singular operator $T : E \to X$ narrow?
- Is every $E$-strictly singular operator $T : E \to X$ narrow?
- Is every $\ell_2$-strictly singular operator $T : E \to X$ narrow?
- Is every $\ell_p$-strictly singular operator $T : E \to X$ narrow?

[Plichko, Popov 1990]
Some known answers
Some known answers

Theorem (Johnson, Maurey, Schechtman, Tzafriri, 1979)

Let $1 < p < 2$ and $T : L_p \rightarrow L_p$. If $T$ is $L_p$-strictly singular, then $T$ is narrow.

Theorem (Bourgain, Rosenthal, 1983)

Let $T : L_1 \rightarrow X$. If $T$ is $\ell_1$-strictly singular, then $T$ is narrow.

Theorem (Rosenthal, 1984)

Let $T : L_1 \rightarrow L_1$. If $T$ is $L_1$-strictly singular, then $T$ is narrow.
Theorem (Flores, Ruiz, 2003)

Every regular $\ell_2$-strictly singular operator $T$ from $L_p$, $1 \leq p < \infty$, to an order continuous Banach lattice is narrow.

Theorem (book of Popov, BR, 2013)

Every regular $\ell_2$-strictly singular operator $T$ from any $q$-concave Banach lattice, $1 \leq q < \infty$, to an order continuous Banach lattice is narrow.

Recall that an operator between Banach lattices is called regular if it is a difference of two positive operators (i.e. operators which map positive elements to positive elements).
**Theorem (Flores, Ruiz, 2003)**

*Every regular $\ell_2$-strictly singular operator $T$ from $L_p$, $1 \leq p < \infty$, to an order continuous Banach lattice is narrow.*

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*Every regular $\ell_2$-strictly singular operator $T$ from any $q$-concave Banach lattice, $1 \leq q < \infty$, to an order continuous Banach lattice is narrow.*

Recall that an operator between Banach lattices is called **regular** if it is a difference of two positive operators (i.e. operators which map positive elements to positive elements).

**Theorem (Mykhaylyuk, Popov, BR, Schechtman, 2012)**

*Every $\ell_2$-strictly singular operator $T$ from $L_p$, $1 < p < \infty$, to any Banach space with an unconditional basis is narrow.*
There are very many open problems remaining in this vein.
Additional open problems

• For a space $E$ of functions, possibly other than a Köthe function space, identify a “small” subclass of functions, such that if an operator $T \in \mathcal{L}(E)$ is not arbitrarily small on functions from the subclass, then there exists an infinite dimensional subspace $F$ of $E$, so that $T|_F$ is an isomorphism.

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- Identify additional properties characterizing narrow operators.
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- Identify additional properties characterizing narrow operators.

**Theorem:** $T \in \mathcal{L}(L_1)$ is narrow iff

$$\lim_{n \to \infty} \| \max_{1 \leq k \leq 2^n} |T \chi_{I_n^k}|_1 = 0,$$

where $I_n^k$ are dyadic intervals.
• Study additional properties of the set of narrow operators.
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Proof techniques

Theorem (Mykhaylyuk, Popov, BR, Schechtman, 2012)

Every $\ell_2$-strictly singular operator $T$ from $L_p$, $1 < p < \infty$, to a Banach space $X$ with an unconditional basis is narrow.

Recall: A Schauder basis $(b_n)_n$ is unconditional if whenever the series $\sum \alpha_n b_n$ converges, it converges unconditionally, that is if there exists a constant $C$ such that for all $n$ and $\varepsilon_k = \pm 1$,

$$\left\| \sum_{k=0}^{n} \varepsilon_k \alpha_k b_k \right\| \leq C \left\| \sum_{k=0}^{n} \alpha_k b_k \right\|$$
We claim that, given an operator with domain $L_p$ which sends some functions to vectors of small norm, there exist also signs which are sent by $T$ to vectors of relatively small norm.

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**geometric** we use fine estimates on $L_p$ norms of functions of special type,
Philosophy

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We use two methods.

**geometric** we use fine estimates on $L_p$ norms of functions of special type,

**probabilistic** we use the martingale structure of the partial sums of the Haar system, stopping times and the central limit theorem.

A similar idea originated in V. Kadets and Schechtman (1992) and was also used in V. Kadets, Kalton and Werner (2005).
Reduction of the form of the operator

Let $h_n$ denote the Haar basis in $L_p$.

**Proposition**

Suppose $1 \leq p < \infty$, $X$ is a Banach space with a basis $(x_n)$, $T : L_p \rightarrow X$ so that

$$\|Tx\| \geq 2\delta$$

for each mean zero sign $x \in L_p$ on $[0, 1]$ and some $\delta > 0$. Then for each $\varepsilon > 0$ there exist an operator $S : L_p \rightarrow X$, a normalized block basis $(u_n)$ of $(x_n)$ and numbers $(a_n)$ so that

1. $Sh_n = a_n u_n$ for each $n \in \mathbb{N}$ with $a_1 = 0$;
2. $\|Sx\| \geq \delta$, for each mean zero sign $x \in L_p$ on $[0, 1]$;
3. $S$ is $\ell_2$-strictly singular, if $T$ is $\ell_2$-strictly singular.
Construction - Probabilistic technique

$S$ is $\ell_2$-strictly singular, so, since Rademachers span an $\ell_2$, $\forall \varepsilon > 0 \ \forall C > 0 \ \exists f$ of the form

$$f = \sum_{m=1}^{N} b_m r_m,$$

so that

$$\|f\|_p = 1, \ \|Sf\|_X < \frac{\delta}{2C},$$

and

$$\max_{m \leq N} |b_m| \leq \frac{\varepsilon}{C}.$$
Flattening of $f$

Fix an $M \in \mathbb{N}$, pick disjoint $\sigma_k \subset \mathbb{N}$ each of size $M$, and put

$$f_k = M^{-1/2} \sum_{n \in \sigma_k} r_n$$

The sequence $\{f_k\}$ is equivalent to an orthonormal basis so the subspace $H$ spanned by the $f_k$-s is isomorphic to a Hilbert space and so an $f$ with $\|Sf\| < \frac{\delta}{2C}$ can be chosen as

$$f = \sum_{k=1}^{K} a_k f_k.$$ 

Since $\|f\| = 1$, the $a_k$ are uniformly bounded, and if $M$ is large enough then the coefficients of $f$ with respect to the Rademacher system, $a_k / \sqrt{M}$ are smaller than $\varepsilon$. 
"Stopping time"-like technique

If $f$ were a sign we would be done, since $\|Sf\| < \frac{\delta}{2C}$ would give a contradiction with $\|Sx\| < \delta$ for all signs.
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If $f$ is not a sign, for each $\omega \in [0, 1]$, we start adding the individual summands forming $Cf(\omega)$, namely $Cb_kh_k(\omega)$, one by one stopping whenever we leave the interval $[-1, 1]$.
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If \( \varepsilon \) is small enough, we do leave this interval for the vast majority of the \( \omega \)-s (by the Central Limit Theorem).
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Whenever we leave the interval, since we just stopped the summation and each \( C|b_k| < \varepsilon \), we get that the absolute value of the stopped sum is 1, to within an error of \( \varepsilon \).

We thus got a function \( g \) which is almost a sign.
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If $f$ were a sign, we would be done, since $\|Sf\| < \frac{\delta}{2C}$ would give a contradiction with $\|Sx\| < \delta$ for all signs.
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Whenever we leave the interval, since we just stopped the summation and each $C|b_k| < \varepsilon$, we get that the absolute value of the stopped sum is 1, to within an error of $\varepsilon$.
We thus got a function $g$ which is almost a sign.
Since $Sh_n = a_n u_n$, where $u_n$ form an unconditional basis,

$$\|Sg\| \leq C\|Sf\| \leq \frac{\delta}{2},$$

and there is a sign $\tilde{g}$ close to $g$, with $\|S\tilde{g}\| < \delta$. **Contradiction**
Let $h_{2^n+i}$ be $L_\infty$ normalized Haar functions, and

$$A = \left\{ \omega \in [0, 1] : \max_{1 \leq 2^m + k \leq 2^{N+1}} \left| C \sum_{2^n+i=2}^{2^m+k} b_n h_{2^n+i}(\omega) \right| > 1 \right\},$$
Let $\bar{h}_{2^n+i}$ be $L_\infty$ normalized Haar functions, and

$$A = \left\{ \omega \in [0, 1] : \max_{1 \leq 2^m+k \leq 2^{N+1}} \left| C \sum_{2^n+i=2}^{2^m+k} b_n \bar{h}_{2^n+i}(\omega) \right| > 1 \right\},$$

Note that as $\varepsilon \downarrow 0$, by the Central Limit Theorem, $f$ converges to a Gaussian random variable so, if $\varepsilon$ is small enough (independently of $C$),

$$\mu([0, 1] \setminus A) \leq \mu\left\{ \omega : \left| \sum_{n=1}^{N} \sum_{i=1}^{2^n} b_n \bar{h}_{2^n+i}(\omega) \right| \leq \frac{1}{C} \right\} \approx \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{C}}^{\frac{1}{C}} e^{-\frac{\omega^2}{2}} d\omega \leq \frac{1}{2C}.$$
We define
\[ \tau(\omega) = \begin{cases} \min \left\{ 2^m + k \leq 2^{N+1} : \left| C \sum_{2^n+i=2}^{2^{m+k}} b_n h_{2n+i}(\omega) \right| > 1 \right\}, & \text{if } \omega \in A \\ 2^N + k, & \text{if } \omega \not\in A \text{ and } \omega \in l^k_N, \end{cases} \]
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\end{cases} \]

and

\[ g(\omega) = C \sum_{2^m+k \leq \tau(\omega)} b_m h_{2^m+k}(\omega). \]
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Since \( S \) sends the Haar functions to functions with disjoint support with respect to the basis \( \{ x_i \} \), by the 1-unconditionality of this basis we get
\[
\| Sg \|_X \leq C \| Sf \|_X < \frac{\delta}{2}.
\]
Let \([0, 1] \setminus A = A_1 \sqcup A_2\), where \(\mu(A_1) = \mu(A_2)\), and define

\[
\tilde{g}(\omega) = \begin{cases} 
\text{sgn}(g(\omega)) & \text{if } \omega \in A \\
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Then \(\tilde{g}\) is a mean zero sign on \([0, 1]\) and

\[
\|g - \tilde{g}\|_p \leq C\varepsilon + \left(1 + \frac{1}{C}\right)\left(\frac{1}{2C}\right)^{1/p},
\]

\(C > 0\) and \(\varepsilon\) so that \(\|S\tilde{g}\|_X < \delta\), which contradicts our assumption about \(S\) and hence \(T\).
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\]

Hence there exist $C > 0$ and $\varepsilon$ so that

\[
\|S\tilde{g}\|_X < \delta,
\]

which contradicts our assumption about $S$ and hence $T$. \hfill \Box
Theorem (Bourgain, Rosenthal, 1983)

Let $T : L_1 \rightarrow X$. If $T$ is $\ell_1$-strictly singular, then $T$ is narrow.
Proof techniques II

Theorem (Bourgain, Rosenthal, 1983)

Let $T : L_1 	o X$. If $T$ is $\ell_1$-strictly singular, then $T$ is narrow.

The main tool is a Ramsey-type result
Lemma

Let S be any set, $0 < a < b < 1$, and $(f_n)_n$ be a sequence of functions on S uniformly bounded by 1. For all $n \in \mathbb{N}$, let

$$A_n = \{ s \in S : |f_n(s)| > b \}, \quad B_n = \{ s \in S : |f_n(s)| < a \}.$$

If for all finite disjoint subsets $I, J$ of $\mathbb{N}$

$$\bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B_j \neq \emptyset,$$

then there exists a subsequence $(f_{n_k})_k \subseteq (f_n)_n$ so that

$$\forall m \in \mathbb{N} \ \forall scalars \ (c_k)_{k=1}^m$$

$$\sup_{s \in S} \left| \sum_{k=1}^{m} c_k f_{n_k}(s) \right| \geq \frac{b - a}{2} \sum_{k=1}^{m} |c_k|$$
Lemma

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$$\sup_{s \in S} \left| \sum_{k=1}^m c_k f_{n_k}(s) \right| \geq \frac{b-a}{2} \sum_{k=1}^m |c_k| = \frac{b-a}{2} \| (c_k)_{k=1}^m \|_1.$$
The lemma is related to a lemma from


see also