Generalized gyrovector spaces and the positive cone of a unital $\mathbb{C}^*$-algebra

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Abstract
The concept of generalized gyrovector space (GGV, for short) is a common generalization of the concept of real normed spaces and of the gyrovector spaces. The addition of a GGV is not necessarily a commutative group but is a gyrocommutative gyrogroup. A typical example of GGV’s is the positive cone of a unital $\mathbb{C}^*$-algebra.

Introduction
The concept of the gyrocommutative gyrogroup is defined by A. A. Ungar. Gyrocommutative gyrogroups are generalized groups, which is not necessarily commutative nor associative. A typical example of the gyrocommutative gyrogroup is the addition of the admissible velocities of the Einstein special relativity. It is called the Einstein gyrogroup. Other typical example is the complex open unit disk $\mathbb{D}$. Define the binary operation on $\mathbb{D}$ by

$$a \circ b = \frac{a + b}{1 - \bar{a}b}$$

for any $a, b \in \mathbb{D}$ then $(\mathbb{D}, \circ)$ is a gyrocommutative gyrogroup. It is motivated by Möbius transformation and is called the Möbius gyrogroup.

Some commutative group admit scalar multiplication and an inner product give rise to real inner product spaces. In full analogy, some gyrocommutative gyrogroups give rise to gyrovector spaces. The concept of the gyrovector space is also defined by A. A. Ungar. It is a generalization of the real inner product space, which addition is not necessarily a commutative group but a gyrocommutative gyrogroup. The Einstein gyrogroup and the Möbius gyrogroup give rise to gyrovector spaces. These are called the Einstein gyrovector space and the Möbius gyrovector space, respectively. A gyrovector space has rich structures as with an inner product space. In the Möbius gyrovector space, its structures are compatible with hyperbolic geometry of the Poincaré disk model.

It is known that the positive cone of a unital $\mathbb{C}^*$-algebra has a gyrocommutative gyrogroup structure. However, the positive cone doesn’t give rise to a gyrovector space. In the joint work with O. Hatori, we define the concept of generalized gyrovector spaces (GGV, for short). It is a common generalization of the concept of the real normed space and of the gyrovector space. The positive cone give rise to a GGV. Its GGV structures are compatible with hyperbolic geometry of the positive cone.

In this paper, we define the concept of the GGV, look the GGV structure of the positive cone of a unital $\mathbb{C}^*$-algebra, and present a Mazur-Ulam type theorem for GGV’s.

Definitions
The concept of gyrocommutative gyrogroup is a generalization of the concept of the commutative group which is not necessarily commutative nor associative. A magma is a set $L$ with a binary operation $L \ast L \to L$. It is known that a magma is a gyrocommutative gyrogroup if and only if it is a K-loop.

Definition 1. A magma $(G, \circ)$ is called a gyrogroup if there exists an element $e$ such that the binary operation $\circ$ satisfies the following (G1) to (G5).

(G1) $\forall a \in G$, $e \circ a = a$. (G2) $\forall a \in G$, $e \circ a = a = a \circ e$. (G3) $\forall a, b, c \in G$, $\exists a \circ (b \circ c) = (a \circ b) \circ c$. (G4) For any $a, b \in G$, the group $(G \circ a, b)$ : $G \to G$ given by $e \to gry[a; b] = eI$, is an automorphism of a magma $(G, \circ)$, which is called the gyroautomorphism of $G$ generated by $a, b \in G$. (G5) $\forall a \in G$, $x \circ (y \circ a) = x \circ (\gamma y a \circ x)$.

A gyrogroup is gyrocommutative if the following (G6) is satisfied.

(G6) $\forall a, b \in G$, $b \circ (a \circ b) = gry[a; b] \circ (a \circ b)$,

Note that $a$ is an identity element of the magma $(G, \circ)$ and $\gamma$ is the inverse element of $a$. (A commutative group is a (gyrocommutative) gyrogroup whose gyroautomorphisms are all trivial.

For study a gyrogroup $(G, \circ)$, it is useful to consider the second binary operation $\oplus \in G$, which is called the coaddition of $(G, \circ)$. If $(G, \circ)$ is a group, it doesn’t distinguish between $\oplus$ and $\circ$.

Definition 2. Let $(G, \circ)$ be a gyrogroup. The gyrogroup coaddition is defined by

$$a \ominus b = gry[a; b](a \circ b)$$

for all $a, b \in G$.

Note that

• the gyrogroup coaddition $\ominus$ is commutative if and only if the gyrogroup $(G, \circ)$ is gyrocommutative.

• if $(G, \circ)$ is a group then $\ominus = \circ$.

Generalized gyrovector spaces
We define the concept of GGV’s. It is a common generalization of the concept of the real normed spaces and of the gyrovector spaces.

Definition 3. Let $(G, \circ)$ be a gyrocommutative gyrogroup with the map $R : G \to G$. Let $\varnothing$ be an injection from $G$ into a real normed space $(V, \parallel \cdot \parallel)$. We say that $(G, \circ, \varnothing)$ is a generalized gyrovector space (GGV, for short) if the it obeys the following axioms:

(V0) $\varnothing(gry[a; b]) = \varnothing(a)$. for any $a, b \in G$;

(V1) $\varnothing(a \ominus b) = \varnothing(a) \ominus \varnothing(b)$. for every $a \in G$;

(V2) $\varnothing(a \circ b) = \varnothing(a) \circ \varnothing(b)$ for any $a, c, r \in G, r \neq 0$;

(V3) $\varnothing(a \circ b) = \varnothing(a \circ r)$. for any $a, c, r \in G, r \neq 0$;

(V4) $\varnothing(a \ominus b) = \varnothing(a) \ominus \varnothing(b)$. for any $a, c \in G$, $\varnothing(c)$ is a generalization of the concept of real normed space of the gyrovector space of $(G, \circ, \varnothing)$. Let $\parallel \cdot \parallel$ be the norm on $G$, which is called the norm of $G(G, \circ, \varnothing)$. We say that $G(G, \circ, \varnothing)$ is a generalized gyrovector space (GGV, for short) if the it obeys the following axioms:

(V0) $\parallel \varnothing(gry[a; b]) \parallel = \parallel \varnothing(a) \parallel$. for any $a, b \in G$;

(V1) $\parallel a \ominus b \parallel = \parallel a \ominus \varnothing(b) \parallel$. for every $a \in G$;

(V2) $\parallel a \circ b \parallel = \parallel a \circ \varnothing(b) \parallel$. for any $a, c, r \in G, r \neq 0$;

(V3) $\parallel a \circ b \parallel = \parallel a \circ r \parallel$. for any $a, c, r \in G, r \neq 0$;

(V4) $\parallel a \ominus b \parallel = \parallel a \ominus \varnothing(b) \parallel$. for any $a, c \in G$.

Note that a real normed space $(V, \parallel \cdot \parallel)$ is a GGV with $\varnothing = \text{id}_V$. On a GGV, we can define the gyroautomorphism. It is an algebraic mid-