**Generalized Bi-circular and 3-circular Projections on Banach Spaces and Related Topics**

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**Introduction**

A projection $P$ on a complex Banach space $X$ is said to be a generalized bi-circular projection (henceforth GBP) if $P + \lambda(I - P) = T$ is an isometry, for some $\lambda \in \mathbb{T} \setminus \{1\}$. Here, $T$ denotes the unit circle in the complex plane. The aim of this poster is to present several results concerning the representation of projections on arbitrary Banach spaces. We also give an example of a GBP which cannot be written as the average of the identity with an isometric reflection, contrary with the standard representation in many different settings. Throughout this poster, $X$ will denote a complex Banach space, and $\Lambda$ a unit modulus complex number not equal to $1$.

**Some Properties of GBP**

- GBPs were introduced by Fošner, Đorđević and Li in 2007.
- GBPs are bi-circular projections, i.e., $(P^2 = I)$ and $(1 - P)^2 = I$.
- The algebra associated with a GBP is surjective.
- GBPs are one of the generalizations of the notion of orthogonal projections from Hilbert spaces to arbitrary Banach spaces.
- Any bi-contractive projection on $C(0,1)$, $L^p$-spaces for $1 \leq p < \infty$, $p \neq 2$ and the space of affine continuous functions on a Choquet simplex $\Lambda(K)$ is a GBP.

**Problems Related to Generalized Bi-circular Projections**

- Characterize generalized bi-circular projections on a given Banach space? In other words, find $\Lambda$?
- Does the set of generalized bi-circular projections on $X$, denoted by $P(X)$, have some interesting properties?
- Whether $P(X)$ is algebraically reflexive?
- The algebraic closure of a subset $B(X)$ is defined as $B^\omega = \{ T \in B(X) \mid \forall x \in X, \exists T_1 \in S$ such that $T(x) = T_1(x)$. $S$ is said to be algebraically reflexive if $S = B^\omega$.
- The isometry group of a finite dimensional Banach space, and $\ell_p$ in $B(\ell_2)$ are algebraically reflexive.
- The isometry group of any infinite dimensional Hilbert space fails to be algebraically reflexive.

**The Scalar $\lambda = 1$**

- If the scalar $\lambda$ associated with a GBP is $-1$, then $P = \frac{I + \overline{\lambda}T}{2}$, where $T^2 = I$.

**A Natural Question**

Whether every GBP on a Banach space is equal to the average of the identity with an isometric reflection. In other words, is $\lambda$ always $-1$? The answer is no as the following example tells us.

**Example**

- Consider $C^3$ with the max norm. Define a projection $P$ on $C^3$ as follows:

$$P(x, y, z) = \frac{1}{2}(x + y, x + y, x + y).$$

- Let $S = P + \lambda(I - P)$ where $\lambda = e^{2\pi i/3}$. This implies that

$$S(x, y, z) = (a + b(y + z), a + b(y + z), a + b(y + z),)$$

with $a = \frac{\lambda + 1}{2\lambda}$ and $b = \frac{1}{2\lambda} - z$.

- $S$ is not an isometry, since $S(0, 0, 1) = (0, 0, 0)$.

- Now, denote $C^n$ so that $S$ becomes an isometric.

$$[x, y, z], \text{max} \{||x, y, z||_1, ||S(x, y, z)||_1, ||S^2(x, y, z)||_1\}.$$  

**Function Spaces**

**Theorem**

Let $P$ be a GBP on $C^4$, then $P + \lambda(I - P)$ is an isometry. Moreover, $P = \frac{I + \overline{\lambda}S}{2}$ where $S$ belongs to the algebra generated by $T$.

**References**


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