Generalized Bicircular Projections on $H^p(U^n)$

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Abstract
A generalized bicircular projection is a projection $P$ such that $P + \lambda(I - P)$ is an isometry for some real $\lambda \neq 1$ with $|\lambda| = 1$. Generalized bicircular projections are of interest because they have been shown to be bi-contractive. The form of the isometries on $H^p(U^n)$ were found by Schneider. We use the form of the the isometries to determine the generalized bicircular projections on the space $H^p(U^n)$.

Introduction
A linear projection $P$ on a Banach space is said to be bi-circular if $P + \lambda(I - P)$ is an isometry for all real $\lambda$. Rather than being an isometry for all real $\lambda$, a generalized bicircular projection (GBP) [4], is given by $P + \lambda(I - P)$ is an isometry for some $\lambda \in T \setminus \{1\}$, where $T$ is the unit circle in $C$. These projections are now referred to as generalized bi-circular projections. Botelho and Jamison [2, 3] and Lin [5] have studied generalized bi-circular projections in various settings. Projections provide information about geometric properties of spaces. Furthermore, GBPs are known to be bi-contractive and as a result they have become an interesting topic of investigation. In many cases, the generalized bi-circular projections have a representation as the average of the identity and an isometric reflection (see [2, 3, 5]). The form of GBPs on $H^p(U^n)$ are found. Unsurprisingly, they are also of the form of the average of the identity and a reflection. Due to this persistent pattern, there is also interest in when the average of two isometries gives a projection (see [1]).

Generalized Bicircular Projections of $H^p(U^n)$

Let $\Pi$ represent a permutation that induces a map on functions of $n$ complex variables by

$$\Pi(f(z_1, z_2, \ldots, z_n)) = f(z_{\Pi(1)}, z_{\Pi(2)}, \ldots, z_{\Pi(n)}).$$

R. B Schneider found the form of the isometries of $H^p(U^n)$ [6].

Theorem 0.1. Suppose $p \neq 2$, $0 < p < \infty$, and $T$ is a linear isometry of $H^p(U^n)$ onto $H^p(U^n)$. Then there is a permutation $\Pi$ such that

$$\Pi - T(f) = \frac{1}{2} \left( \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} \right) \overline{\overline{f}}(z_{\Pi(1)}, z_{\Pi(2)}, \ldots, z_{\Pi(n)})$$

where the $\phi_i$ are conformal maps of the unit disc onto itself and $b$ is a unimodular complex number. Conversely, (1) defines a linear isometry of $H^p(U^n)$ onto $H^p(U^n)$.

A generalized bi-circular projection is a projection $P$, such that for some $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$, $P + \lambda(I - P)$ is an isometry.

Theorem 0.2. $P$ is a generalized bicircular projection on $H^p(U^n)$ if and only if $P$ is trivial or

$$P(f(z_1, z_2, \ldots, z_n)) = \frac{1}{2} \left( \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} \right) \overline{\overline{f}}(z_{\Pi(1)}, z_{\Pi(2)}, \ldots, z_{\Pi(n)}) + f(z_1, z_2, \ldots, z_n)$$

where each of the $\phi_i$ are conformal maps of the unit disk onto itself such that $\phi_{\Pi(i)} \circ \phi_{\Pi(i)}(z_{\Pi(i)}) = z_{\Pi(i)}$ and $\Pi$ is a permutation that induces a map on functions of complex variables.

Sketch of Proof
Let $T$ be an isometry. Then $P + \lambda(I - P) = T$. Solving for $P$ gives that $P = \frac{T - \lambda I}{1 - \lambda}$. Since $P$ is a projection, $P^2 = P$, thus $(T - \lambda I)^2 = (1 - \lambda)(T - \lambda I)$. This then can be reduced to

$$T^2 - (\lambda + 1)T + \lambda I = 0$$

Then

$$Tf(z_1, \ldots, z_n) = b((\phi_1(z_1), \ldots, \phi_n(z_n)))^{1/p} f(\phi_1(z_1), \ldots, \phi_n(z_n))$$

and

$$T^2f = b((\phi_1(z_1), \ldots, \phi_n(z_n)))^{1/p} \cdot (\phi_1(z_1), \ldots, \phi_n(z_n)) f(\phi_1(z_1), \ldots, \phi_n(z_n))$$

This makes (2):

$$b((\phi_1(z_1), \ldots, \phi_n(z_n)))^{1/p} \cdot (\phi_1(z_1), \ldots, \phi_n(z_n)) = \lambda f(z_1, \ldots, z_n)$$

Since this equation must hold for all functions $f(z_1, \ldots, z_n)$, consider $f(z_1, \ldots, z_n) = 1$ which reduces (3) to

$$\lambda f(z_1, \ldots, z_n) = \lambda f(z_1, \ldots, z_n)$$

This makes (2):

$$\lambda f(z_1, \ldots, z_n) = \lambda f(z_1, \ldots, z_n)$$

References

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