CLASSIFYING FUNCTIONS IN THE KERNEL OF THE ADJOUT OF A COMPOSITION OPERATOR ON THE HARDY SPACE

Brittney Miller  Department of Mathematics, Purdue University

ABSTRACT

Let \( \varphi \) be an analytic function from \( \mathbb{D} \) to itself. Then, the composition operator \( C_\varphi \), with symbol \( \varphi \), is defined by \( C_\varphi f = f \circ \varphi \) for \( f \) in a Hilbert space of analytic functions on \( \mathbb{D} \). In 2008, Hammond, Moorhouse, and Robbins gave an explicit formula for the adjoint \( C_\varphi^* \) in the Hardy space. If \( \varphi \) is not univalent, the kernel of \( C_\varphi^* \) is infinite-dimensional. With this post, I will illustrate how their formula leads to a classification of functions in \( \ker(C_\varphi^*) \) for certain classes of symbols \( \varphi \).

MAIN RESULTS

Theorem 1 (Hammond-Moorhouse-Robbins, 2008 [3]). Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be a non-constant rational map, and let \( C_\varphi \) act on \( H^2(\mathbb{D}) \). Set \( \sigma(z) = \frac{1}{\varphi^{-1}(1/2)} \), \( \psi(z) = \frac{\varphi'(z)}{\sigma(z)} \), and \( \varphi(\infty) = \lim_{|z| \to \infty} \varphi(z) \). Then,

\[
(C_\varphi^* f)(z) = \frac{f(0)}{1 - \varphi(\infty)z} + \sum \psi(z) f(\sigma(z)),
\]
where the sum is taken over the branches of \( \sigma \).

Theorem 2 (M., 2015). Let \( \varphi \) be a rational map of degree two mapping \( \mathbb{D} \to \mathbb{D} \) with \( \varphi(z) = \frac{az^2 + bz + c}{dz^2 + ez + h} \) where \( h \neq 0 \). Let \( \sigma_1 \) and \( \sigma_2 \) be the branches of \( \sigma \) from Theorem 1. There exists a function \( \zeta \), not the identity, such that

\[
\zeta(z) = \frac{1}{\psi^{-1}(\varphi(1/2))} \quad \text{and} \quad \zeta(\sigma_1) = \sigma_2.
\]

Furthermore,

\[
\zeta(z) = - \frac{(\overline{\alpha} - \overline{a}) + (\overline{\alpha h} - \overline{a})z}{(\overline{a} - \overline{a}) + (\overline{h} - \overline{a})z}.
\]

Theorem 3 (M., 2015). Let \( \varphi \) be a rational map of degree two mapping \( \mathbb{D} \to \mathbb{D} \) with \( \varphi(z) = \frac{az^2 + bz + c}{dz^2 + ez + h} \) where \( h \neq 0 \), and let \( C_\varphi \) act on \( H^2(\mathbb{D}) \). Then, \( f \in \ker(C_\varphi^*) \) if and only if

\[
(\zeta(f)(z) + \zeta'(z) f(\zeta(z))) = 0.
\]

Example 1 (Non-univalent Symbol). Let \( \varphi(z) = \frac{z^2 + 1}{2} \). Let \( \sigma_1 \) and \( \sigma_2 \) be the branches of \( \sigma \) with their respective maps \( \psi_1 \) and \( \psi_2 \) given in Theorem 1. Then,

\[
\ker(C_\varphi^*) = \{ f \in H^2(\mathbb{D}) \mid f(z) = f(-z) \}.
\]

\[
\sigma_1(z) = \frac{\sqrt{z}}{2 - z} \quad \sigma_2(z) = -\frac{\sqrt{z}}{2 - z}
\]
\[
\psi_1(z) = \frac{1}{2 - z} \quad \psi_2(z) = \frac{1}{2 - z}
\]
\[
\zeta(z) = -z.
\]

The image of \( \varphi \)

TRIVIAL CASES?

Example 2 (Univalent Symbol)

Let \( \varphi(z) = \frac{1}{20} (z^2 + 10z + 9) \).

\( \varphi \) is a weak-star generator of \( H^\infty \) and so the range of \( C_\varphi \) is dense in \( H^2(\mathbb{D}) \) [4].

Therefore, the kernel of \( C_\varphi^* \) is \{0\}.

While Theorem 3 gives a classification of functions in \( \ker(C_\varphi^*) \) for every rational symbol of degree two mapping \( \mathbb{D} \to \mathbb{D} \), it does not specify for which symbols \( \ker(C_\varphi^*) = \{0\} \). This example suggests that \( C_\varphi^* \) with univalent symbol may have trivial kernel.

FUTURE WORK

Classify \( \ker(C_\varphi^*) \) for all rational symbols \( \varphi \) in the Hardy space.

If possible, give a classification of univalent symbols \( \varphi \) that have non-trivial \( \ker(C_\varphi^*) \).

Extend these results about \( \ker(C_\varphi^*) \) in the Hardy space to \( \ker(C_\varphi^*) \) in the Bergman space.

REFERENCES


