An Example Of A Fuzzy Normed Space Which Is Not A Normed Linear Space In The Classical Sense.

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Abstract
The purpose of this paper is to give an example of a Felbin’s and Samanta’s type fuzzy normed linear space, such that the fuzzy version remains true, while it fails to be correct in classical analysis. This shows how the category of fuzzy normed linear spaces differs from the classical normed linear spaces in general.

Introduction
In 1984, Katsaras first introduced the idea of fuzzy norm on a linear space. In 1992, Felbin introduced an idea of a fuzzy norm on a linear space whose associated metric is Kalka type. In 2003, T. Bag and S.K. Samanta introduced a definition of fuzzy norm and proved the decomposition theorem of fuzzy norm to a family of crisp norms. In [4, 5] the authors have shown that a fuzzy normed linear space is a topological vector space in the sense of classical analysis, also continuing in fuzzy analysis and classic analysis are equivalent. So all results and theorems of topological vector spaces apply to fuzzy normed linear spaces in general. So studying the fuzzy type of the above theorems help us to improve the fuzzy version of operator theory which is of great importance. As a new result, we show that if $C(\Omega)$ is a fuzzy normed space, while it is not classical normed when is a nonempty open subset in some euclidean space. So if we replace the normed space by a fuzzy normed space $C(\Omega)$, then we get an important result in the fuzzy normed linear spaces, while the same statement does not hold true in classical analysis.

Preliminaries
Definition 1. A mapping $\gamma : R \rightarrow [0, 1]$ is called a fuzzy real number, whose $\alpha$-level set is denoted by $[\gamma]_{\alpha} = \{x \in R | \gamma(x) \geq \alpha\}$, if it satisfies two axioms:

(N1) There exists $t_0 \in R$ such that $\gamma(t_0) = 1$.
(N2) For each $\alpha \in (0, 1]$, $[\gamma]_{\alpha} = [\gamma]_{\alpha'} \subseteq [\gamma]_{\alpha''}$, $\alpha' < \alpha'' < +\infty$.

The set of all fuzzy real numbers denoted by $B(I)$. If $\gamma \in B(I)$ and $\gamma(t) = 0$ whenever $t < 0$, then $\gamma$ is called a non-negative fuzzy real number and $R^+(I)$ stands for the set of all non-negative fuzzy real numbers.

Theorem 1. Let $X$ be a fuzzy linear vector space satisfying $\langle N \rangle$. Then $X$ is a Hausdorff topological vector space, whose neighborhood base of origin $0$ is $\{N(x, \alpha) | x \in X, \alpha \in (0, 1] \}$.

Theorem 2. Let $X$ be a fuzzy normed linear space satisfying $\langle N \rangle$. Then $X$ is a Hausdorff topological vector space, whose neighborhood base of origin $0$ is $\{N(x, \alpha, t) | x \in X, \alpha \in (0, 1], t > 0 \}$.

Theorem 3. Let $T : X \rightarrow Y$ be a mapping where $X$ and $Y$ are fuzzy normed spaces. Then $T$ is continuous iff $T(N(x, \alpha)) \subseteq N(T(x), \alpha)$. Moreover, $T$ is continuous on $X$.

Theorem 4. Let $T : X \rightarrow Y$ be a linear operator, where $X$ and $Y$ are fuzzy normed linear spaces (Felbin’s type or Samanta’s type). $T$ is fuzzy continuous on $T$ is fuzzy bounded.

Main Results
Definition 2. Let $X$ be a vector space. A fuzzy subset $N$ of $X \times R$ is called a fuzzy norm on $N$ if the following conditions:

\begin{align*}
N(x, t) = \{z \in X | t \geq (x, t) \} \quad & \forall \alpha \in [0, 1] \quad \forall \beta \in [0, 1] \\
N(x, 0) = \{z \in X | (x, t) = 0 \} \\
N(x, t) \supseteq N(z, t) \supseteq N(x, t - \alpha) \\
N(x, 0) = \{z \in X | (x, t) = 0 \} \\
\end{align*}

The pair $(X, N)$ is called a fuzzy normed space.

Theorem 5. Let $(X, N_1)$ and $(Y, N_2)$ be mappings, where $(X, N_1)$ and $(Y, N_2)$ are fuzzy normed linear spaces. Then $T$ is said to be fuzzy bounded on $C(\Omega)$ if for each $C(\Omega)$ bounded subset of $X$ and for all $\alpha > 0$, there exists $C(\Omega)$ such that $T(x) \subseteq T(x)$.

Main Objectives
Let $f \in C(\Omega)$ define $f_{\alpha} = \sup \{f(x) | x \in K_{\alpha} \}$. Then $f_{\alpha} = f_{\alpha}$. Also define $p_0(f, t) = \sup \{f(x) | x \in K_{\alpha} \}.$

Let $f$ be a fuzzy seminorm on $C(\Omega)$ and $p_0(f, t)$ be a fuzzy seminorm for all $\alpha \in (0, 1]$. Then $C(\Omega)$ is a fuzzy normed space. But $C(\Omega)$ is not normable in classical analysis, where $\Omega$ is an open subset of $R^n$.

Conclusions
The category of normed spaces can be embedded in the category of fuzzy normed spaces.

References