Definition 1 (elementary Lipschitz tensors and Lipschitz tensor product) For \((x, y) \in \mathbb{X}^2\) and \(e \in E\), the elementary Lipschitz tensor \(\alpha_{(x, y)}(e) \in \text{Lip}(X, E^*) \rightarrow E\) is the bounded linear functional defined by
\[
\beta(e, x, y) = \text{Lip}(e)(\alpha_{(x, y)}(e)) = f(x(e)) - f(y(e)).
\]

The Lipschitz tensor product \(\mathbb{X} \times E \rightarrow E\) is defined as the vector subspace of \(\text{Lip}(X, E^*)^*\) spanned by the set \(\{\alpha_{(x, y)}(e) \mid e \in E\}\).

Each element \(u \in \mathbb{X} \times E\) can be represented as \(u = \sum_{i=1}^{n} \alpha_{(x_i, y_i)}(e_i)\).

Theorem 2
(i) If \((X, E, \text{Lip}(X, E^*))\) is a dualizable Lipschitz cross-norm on \(X\), then \(\text{Lip}(X, E)\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(X, E^*))\).

(ii) The set \(\text{Lip}(X, E)\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(X, E^*))\).

Proposition 5
If \(u\) is a dualizable Lipschitz cross-norm on \(X\), then \(\text{Lip}(X, E)\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(X, E^*))\).

Definition 6
(i) The Lipschitz injective norm \(\text{Lip}(X, E)\) is established, for each \(u \in \mathbb{X} \times E\), through the formula
\[
\text{Lip}(u)(f) = \sup_{x, y \in \mathbb{X}, e \in E} \frac{f(x(e)) - f(y(e))}{\beta(e, x, y)}.
\]

(ii) The Lipschitz projective norm \(\text{Lip}(X, E)\) is defined by the formula
\[
\text{Lip}(u)(f) = \sup_{x, y \in \mathbb{X}, e \in E} \frac{f(x(e)) - f(y(e))}{\beta(e, x, y)}.
\]

(iii) If \(u\) is a dualizable Lipschitz cross-norm on \(X\), then \(\text{Lip}(u)(f)\) is defined by the formula
\[
\text{Lip}(u)(f) = \sup_{x, y \in \mathbb{X}, e \in E} \frac{f(x(e)) - f(y(e))}{\beta(e, x, y)}.
\]

References
[1] J. A. Chávez-Domínguez, Joint work with M. G. Cabrera Padilla, A. Jiménez Vargas (University of Almería) and J. A. Chávez-Dominguez (University of Texas at Austin).

Lipschitz tensor product
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Abstract
We introduce the notion of a Lipschitz tensor product \(X \times E\) of a pointed metric space \(X\) and a Banach space \(E\). To ensure the good behavior of a norm on \(X \times E\), we consider the concept of dualizable (respectively, uniform) Lipschitz cross-norm on \(X \times E\) and define some examples of such norms are studied. Using Lipschitz cross-norms, we present a new class of operators, the cross-norm-Lipschitz operators. Our main theorem states that the space \(\text{Lip}(X, E^*)\) of all \(\alpha\)-Lipschitz operators is isometrically isomorphic to the dual space \(X^* \times E\). From this theorem, a first consequence is obtained for the space of \(p\)-summing operators introduced by Farner and Johnson. Finally, the notion of cross-norm-Lipschitz approximable operator is defined in a natural way, and we see its relationship with the associated Lipschitz tensor product.

Duality for cross-norm-Lipschitz operators

Definition 8 (cross-norm-Lipschitz operators)
(i) A function \(f \in \text{Lip}(X, E)\) is said to be a \(\alpha\)-Lipschitz operator if there exists a real constant \(C \geq 0\) such that \(|\|f(x) - f(y)\|_{\text{Lip}}| \leq C\|x - y\|_{\text{Lip}}\) for all \(x, y \in X\). In that case, the infimum of such constants \(C\) is denoted by \(\text{Lip}(f)\) and called the \(\alpha\)-Lipschitz norm of \(f\).

(ii) Let \(\alpha\) be a \(\alpha\)-Lipschitz operator from \(X\) into \(E\) and it is a normed space with the \(\alpha\)-Lipschitz norm \(\text{Lip}(\alpha)\).

Definition 9 (main theorem)
Let \(\alpha\) be a \(\alpha\)-Lipschitz cross-norm on \(X \times E\). Then \(\text{Lip}(X, E)\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(X, E^*))\) via the map \(\text{Lip}(X, E^*) \rightarrow (X_{\infty}, \text{Lip}(X, E^*))\).

Moreover, its inverse \((X_{\infty}, \text{Lip}(X, E^*)) \rightarrow \text{Lip}(X, E)^*\) is given by \(\text{Lip}(X, E)^* \rightarrow (X_{\infty}, \text{Lip}(X, E^*))\). For all \(x \in X\) and \(\psi \in (X_{\infty}, \text{Lip}(X, E^*))\), let \(\text{Lip}(X, E)\) denote the natural norm agrees with \((\text{Lip}(X, E), \text{Lip}(\alpha))\). As a consequence, \(\text{Lip}(X, E)^*\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(\alpha)^*)\).

Farner and Johnson introduced \(\text{Lip}(\alpha)\) in [3] of the notion of \(p\)-summing operators between metric spaces for \(1 \leq p \leq \infty\) (see [1] for the case \(p = \infty\)). We see that Lipschitz \(p\)-summing operators between \(X\) and \(E\) are a particular case of cross-norm-Lipschitz operators.

Theorem 10
Let \(1 \leq p \leq \infty\) and \(\psi_x, \psi_y\) be the conjugate index of \(p\). Then the Banach space \(L^p(X_{\infty}, E)\) of all \(\alpha\)-Lipschitz \(p\)-summing operators from \(X\) into \(E\) endowed with its natural norm agrees with \((L^p(X_{\infty}, E)), \text{Lip}(\alpha))\). As a consequence, \(L^p(X_{\infty}, E)^*\) is isometrically isomorphic to \((X_{\infty}, L^p_{\infty}(X_{\infty}, E^*))\).

Cross-norm-Lipschitz approximable operators

Definition 11 (cross-norm-Lipschitz approximable operators)
Let \(\alpha\) be a \(\alpha\)-Lipschitz cross-norm on \(X \times E\). A \(\alpha\)-Lipschitz operator \(f \in \text{Lip}(X, E)\) is said to be \(\alpha\)-Lipschitz approximable if \(f\) is the limit of the \(\alpha\)-Lipschitz norm \(\text{Lip}(\alpha)\) of a sequence of \(\alpha\)-Lipschitz finite-rank operators from \(X\) to \(E\).

Theorem 12
Let \(\alpha\) be a \(\alpha\)-Lipschitz cross-norm on \(X \times E\). Let \(u\) be a \(\alpha\)-Lipschitz cross-norm on \(X \times E\) and let \(\text{Lip}(u)(f)\) be its associated norm (Proposition 5). Then \(\text{Lip}(u)(f)\) is isometrically isomorphic to \((X_{\infty}, \text{Lip}(u)^*)\).

As a consequence, the space of all \(\alpha\)-Lipschitz approximable operators from \(X\) to \(E\) is isometrically isomorphic to the completion of \((X_{\infty}, \text{Lip}(\alpha)^*)\).

References