Masters Comprehensive exam
Complex Analysis

February 2013

Answer three of the following five questions.

1. Prove or disprove the existence of an analytic function $f$ defined on $\{ z : |z| < 2 \}$ such that
   
   $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}, \quad n = 1, 2, 3, \ldots$

2. (a) State Rouché’s Theorem.
    (b) Prove or disprove the existence of a sequence of analytic functions $f_n$ defined on $\{ z : |z| < 2 \}$ such that $f_n(z) \to \pi$ uniformly on the circle $\{ z : |z| = 1 \}$?

3. Evaluate the integral
   
   $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx$.

4. Find a meromorphic function $f$ with simple poles at $z = \sqrt{n}, n = 1, 2, 3, \ldots$, with residues $\text{res}_{z=\sqrt{n}}(f) = 1$.

5. (a) State the Maximum Modulus Principle.
    (b) Suppose $|z_i| = 1$ for $i = 1, 2, \ldots, n$. Prove that $\prod_{i=1}^{n} |z - z_i| > 1$ for some $z$ with $|z| = 1$. 

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