Masters’ Comprehensive Exam: Topology

September 20, 2008

Do any three of the following four problems. You have 1 hour.

1. (a) Prove or disprove that the subspace topology on the set of integers, \( \mathbb{Z} \), in \( \mathbb{R} \) (with its usual topology) is the same as the discrete topology on \( \mathbb{Z} \).
(b) Prove or disprove that the subspace topology on the set of rationals, \( \mathbb{Q} \), in \( \mathbb{R} \) (with its usual topology) is the same as the discrete topology on \( \mathbb{Q} \).

2. Let \( X \) be a metric space. Show that the following are equivalent:
   (a) \( X \) has a countable basis;
   (b) \( X \) is Lindelöf (i.e., every open cover of \( X \) has a countable subcover);
   (c) \( X \) has a countable dense subset.

3. Suppose that \( A \times B \subseteq U \subseteq H \times K \), where \( A, B, H, \) and \( K \) are compact Hausdorff spaces and \( U \) is open in the product space \( H \times K \). Show that there exist open subsets \( V \) of \( H \) and \( W \) of \( K \) such that \( A \times B \subseteq V \times W \subseteq U \).

4. Let \( f : [a, b] \to \mathbb{R} \) be a real-valued function on a closed interval and let \( G = \{(x, f(x)) \mid x \in [a, b]\} \subseteq \mathbb{R}^2 \) be its graph. Prove or give a counterexample for the following.
   (a) If \( G \) is connected, then \( f \) is continuous.
   (b) If \( f \) is continuous, then \( G \) is connected.