Master’s Exam: Core Topics

Spring 2003

Answer any Six Questions; credit will be given for the best six questions
You must state clearly any general results you use. Explain your answers and show all working

1. Let \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) be sequences of real numbers. For each of the following statements, prove it or give a counterexample:
   (a) If \(\sum_{n=1}^\infty a_n\) exists then \(\lim_{n \to \infty} a_n = 0\).
   (b) If \(\lim_{n \to \infty} a_n = 0\) then \(\sum_{n=1}^\infty a_n\) exists.
   (c) If \(\sum_{n=1}^\infty a_n\) and \(\sum_{n=1}^\infty b_n\) exist then \(\sum_{n=1}^\infty (a_n + b_n)\) exists.

2. Prove direct from the definition that the function \(f(x) = \frac{1}{x+1}\) is continuous at each point \(a \in \mathbb{R} \setminus \{-1\}\).

3. Consider the real-valued function defined as follows:
\[
f : (0, \infty) \to \mathbb{R} \\
x \to \frac{1}{x}
\]
   (a) Show that \(f\) is not uniformly continuous.
   (b) Characterize those subsets \(A \subseteq (0, \infty)\) such that \(f\) restricted to \(A\) is uniformly continuous.

4. Consider the reals with the usual topology and \(A = (0, 1] \cup \{1 + \frac{1}{n}\}_{n=1,2,...}\) with the induced topology.
   (a) Is the interval \((0, 1]\) a closed subset of \(A\)?
   (b) Determine the interior and the closure, in \(A\), of \(\{1 + \frac{1}{n}\}_{n=1,2,...}\).
5. (a) Let $G$ be a finite group and let $H$ be a subgroup of $G$ of index 2. Show that $H$ is a normal subgroup of $G$.

(b) Give an example of a group $G$ and a subgroup $H$ of index 3 that is not normal.

6. Let $G$ be a group and let $H$ be the subgroup generated by all expressions of the form $xyx^{-1}y^{-1}$.

(a) Show that $H$ is a normal subgroup of $G$.

(b) Show that the quotient $G/H$ is abelian.

7. Let $U$ and $V$ be two real vector spaces and let $\text{Hom}(U, V)$ be the set of linear maps from $U$ to $V$.

(a) Show that $\text{Hom}(U, V)$ is a real vector space under pointwise addition and scalar multiplication, $(f + g)(x) = f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$.

(b) Show that $\dim \text{Hom}(U, V) = (\dim U)(\dim V)$.

8. Let $T: V \to V$ be a linear map on a real vector space $V$ such that $T^2 = 1$.

(a) Show that $V_+ = \{ x : Tx = x \}$ and $V_- = \{ x : Tx = -x \}$ are subspaces of $V$.

(b) Show that $V$ is isomorphic to the direct sum of $V_+$ and $V_-$. 