Master’s Exam: Basic Areas

March 13th 2004

Answer any five questions.
Show all working; State clearly all theorems that you apply.

1. Define a sequence \((x_n)\) by \(x_0 = 1, x_{n+1} = (x_n/2) + \sqrt{x_n}\). Prove that the sequence \((x_n)\) is convergent and identify the limit.

2. (a) State and prove the comparison test for convergence of series.
(b) Prove that the series
\[
\sum_{n=2}^{\infty} \frac{n + 5\sqrt{n}}{n^{5/2} - 1}
\]
is convergent.

3. Define for \(x > 0\), \(L(x) = \int_{1}^{x} \frac{1}{t} \, dt\). Without assuming any properties of logarithms, show that
   (a) \(L(x)\) is a strictly increasing function, i.e., \(x_1 < x_2\) implies \(L(x_1) < L(x_2)\).
   (b) \(L(xy) = L(x) + L(y)\) for \(x, y > 0\).
   (c) \(L(x)\) tends to \(\infty\) as \(x \to \infty\).

4. (a) Define what it means for a subset of a metric space to be compact.
(b) Show that the image of a continuous function on a compact set is compact.
(c) If \(C\) is a compact subset of a metric space \(X\), show that for any point \(x \in X\) there is a point \(y \in C\) such that the distance \(d(x, y)\) is minimal, i.e., \(d(x, y) \leq d(x, z)\) for all \(z \in C\).

Turn over
5. Let \( U_i \) be subspaces of a finite dimensional vector space \( V \).
   (a) Show that \( \dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) \).
   (b) Give an example that shows that \( \dim(U_1 + U_2 + U_3) + \dim(U_1 \cap U_2 \cap U_3) \) need not be equal to \( \dim(U_1) + \dim(U_2) + \dim(U_3) \) in general.

6. The normalizer \( N_H \) of a subgroup \( H \) of a group \( G \) is defined as \( \{ g \in G : gH = Hg \} \).
   (a) Prove that \( H \) is a normal subgroup of \( N_H \).
   (b) By considering a suitable subgroup of the dihedral group with 6 elements, show that \( N_H \) need not be a normal subgroup of \( G \).

7. (a) Suppose \( G \) is a group in which \( x^5 = 1 \) for all \( x \in G \). Show that either \( G = \{1\} \) or the order of \( G \) is divisible by 5.
   (b) Give an example of a group \( G \neq \{1\} \) in which \( x^6 = 1 \) for all \( x \in G \), but the order of \( G \) is not divisible by 6.