Masters’ Comprehensive Exam: Core Topics

February 9, 2008

*Answer any five of the following seven problems.*

1. Define a sequence \( (x_n)_{n=0}^{\infty} \) by \( x_0 = 2 \) and \( x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \) for \( n \geq 0 \).
   (a) Show that \( x_n \) converges as \( n \to \infty \).
   (b) Identify the limit of the sequence \( (x_n)_{n=0}^{\infty} \).

2. Prove or disprove:
   (a) The product of two uniformly continuous functions on \( \mathbb{R} \) is also uniformly continuous.
   (b) The product of two uniformly continuous functions on \([0, 1]\) is also uniformly continuous.

3. Let \( (x_n)_{n=0}^{\infty} \) be a sequence of real numbers. Prove that the following are equivalent.
   (a) \( \lim_{n \to \infty} x_n = a \).
   (b) Every subsequence of \( (x_n)_{n=0}^{\infty} \) contains a subsequence that converges to \( a \).

4. Prove that a compact subset of a Hausdorff space is closed.

5. Determine which of the following groups are isomorphic.
   (i) \( \mathbb{Z}_2 \times \mathbb{Z}_4 \), (ii) \( \mathbb{Z}_8 \), (iii) \( \mathbb{Z}_2 \times \mathbb{Z}_3 \), (iv) \( \mathbb{Z}_6 \).

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6. Let $G$ be the group of $2 \times 2$ invertible upper triangular real matrices under matrix multiplication, i.e., matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}
\]
with $a, b, d \in \mathbb{R}$, $ad \neq 0$.

(a) Show that the set $K$ of matrices in $G$ that are of the form \[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\] form a normal subgroup of $G$.

(b) Show that the quotient group $G/K$ is isomorphic to $\mathbb{R}^\times \times \mathbb{R}^\times$.

7. Let $T: V \to V$ be a linear map on a real vector space $V$ such that $T^2 = 1$.

(a) Show that $V_+ = \{x \mid Tx = x\}$ and $V_- = \{x \mid Tx = -x\}$ are subspaces of $V$.

(b) Show that $V$ is isomorphic to the direct sum of $V_+$ and $V_-$. 