1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

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<th>Problem</th>
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2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.
1. Let $X_1, X_2$ be two independent random variables from $U(0, 1)$.

(a) Find the joint p.d.f. of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.
(b) Find the marginal p.d.f. of $Y_1$.
(c) Find the marginal p.d.f. of $Y_2$. 
2. Let $X$ be a random variable with the p.d.f.

$$f(x; \theta) = \theta (1 - x)^{\theta - 1}, \quad 0 < x < 1, 0 < \theta < \infty.$$ 

(a) Find the p.d.f. of $Y = -\ln(1 - X)$.
(b) Find the p.d.f. of $Y = \ln(X/(1 - X))$. 
3. Let $X_1, X_2, \cdots, X_n$ be a random sample taken from the distribution with the p.d.f.

$$f(x; \theta) = \theta x^{\theta - 1}, \quad 0 < x < 1, \quad 0 < \theta < \infty.$$ 

(a) Derive the moment estimator of $\theta$.

(b) Derive the maximum likelihood estimator of $\theta$. 
4. Suppose $X_1, X_2, X_3, \ldots X_{32}$ be a random sample with a beta distribution with the p.d.f. 
\[ f(x) = 2x, \quad 0 < x < 1. \]

(a) Let $Y$ be the number of these random variables ($X_i, i = 1, 2, \ldots, 32$) whose values less than 0.5. Find $P(Y \leq 10)$.

(b) Approximate $P(\sum_{i=1}^{14} X_i > \sum_{i=15}^{32} X_i)$. 
5. Let $X$ be a random variable whose probability mass function under $H_0$ and $H_1$ is given by

<table>
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<th>$x$</th>
<th>1</th>
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<tr>
<td>$f(x</td>
<td>H_0)$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
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<tr>
<td>$f(x</td>
<td>H_1)$</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
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(a) Use the Neyman-Pearson Lemma to find the most powerful test for $H_0$ vs. $H_1$ with size $\alpha = 0.05$.

(b) Compute the probability of Type II error for the above test.
6. Let $X_1$ and $X_2$ be independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), i = 1, 2$. Let $Y = X_1 + X_2$ and $Z = X_1X_2$

(a) Find the distribution of $X_1$ given $Y = y$.

(b) Find $\text{Cov}(Y, Z)$

(c) If $\mu_1 = \mu_2 = 0$, are $Y$ and $Z$ independent? (Fully explain your answer.)
7. Let \( X_{i1}, \ldots, X_{in_i}, i = 1, 2 \) be independent random samples from exponential distributions with means \( \theta_i, i = 1, 2 \).

(a) Derive a likelihood ratio test for testing \( H_0 : \theta_1 = \theta_2 \) versus \( H_1 : \theta_1 \neq \theta_2 \)

(b) Show that the test statistic can be expressed as

\[
T = \frac{\sum_{j=1}^{n_1} X_{1j}}{\sum_{j=1}^{n_1} X_{1j} + \sum_{j=1}^{n_2} X_{2j}}
\]

(c) Find the distribution of \( T \) when \( H_0 \) is true.
8. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x; \theta) = 3\theta x^2 e^{-\theta x^3}, \quad x > 0.$$ 

(a) Let $M$ be the median of this distribution. Find the MLE of $M$.

(b) Suppose we put a Gamma($\alpha, \beta$) prior distribution on $\theta$, find the Bayes estimate of $\theta$ using squared error loss function.
9. Let \( \{X_1, \ldots, X_n\} \) be independently and identically distributed normal variables with mean 0 and variance 1. Put \( Y_1 = \frac{1}{n} \sum_{i=1}^{n} X_i \) and \( Y_2 = \sum_{i=1}^{n} (X_i - Y_1)^2 \).

(a) Show that \( Y_1 \) and \( Y_2 \) are independently distributed of each other stochastically.
(b) What is the sampling distribution of \( Y_2 \)?
10. Let \( \{X_1, \ldots, X_n\} \) be a random sample from the density 
\[
f(x; \theta) = e^{-\theta x} / x!, \quad x = 0, 1, \ldots, \infty, \quad \theta > 0.
\]

(a) Obtain a sufficient and complete statistic for \( \theta \).
(b) Derive the UMVUE (Uniformly Minimum Variance Unbiased estimator) of 
\( P\{X = 1\} \).
11. Let \( \{X_1, \ldots, X_n\} \) be a random sample from the density \( f(x, \theta) = \theta^{-1}e^{-x/\theta}, 0 < x; f(x, \theta) = 0, \) if \( x \leq 0. (\theta > 0) \)

(a) Derive the size-\( \alpha \) UMP (Uniformly Most Powerful) test for testing \( H_0 : \theta = 1 \) vs \( H_1 : \theta > 1. \)

(b) Assume that the observed sample mean is 0.98 and \( n=20. \) Based on this observed data, obtain the p-value of your test. From on this analysis, what conclusions you will make?
12. Let \( \{X_1, \ldots, X_{10}\} \) be a random sample from the normal density with mean \( \mu \) and variance 1. Let the prior distribution of \( \mu \) be given by \( P(\mu) \propto e^{-\frac{1}{2}(\mu-2)^2} \), \( \mu \) real.

Assume that the observed value of the sample mean is 1.5. Derive the 95% HPD (Highest Posterior Density) interval of \( \mu \).