Answer any six of the following eight questions. You must state clearly any general results you use.

1. Prove that if $G$ is a non-trivial $p$-group then the center of $G$ is non-trivial. Deduce that every $p$-group is solvable.

2. Prove that if $G$ is a simple group of order 168 then $G$ contains exactly 48 elements of order 7.

3. Let $R$ be a ring with 1. Show that if $x$ is contained in every maximal ideal of $R$ then $1 + x$ is a unit.

4. Let $R$ be a commutative ring with 1.
   (a) Show that every maximal ideal is prime.
   (b) Show that if $R$ is a PID then every non-zero prime ideal is maximal.

5. Let $K = \mathbb{Q}(t)$ be the field of rational functions in the indeterminate $t$. Let $\phi: \mathbb{Q}(t) \to \mathbb{Q}(t)$ be the field homomorphism which fixes $\mathbb{Q}$ and sends $t$ to $\frac{2}{7}t$.
   (a) Show that $\phi$ is an automorphism of $K$ of order 2.
   (b) Show that the fixed field of $G = \{1, \phi\}$ is $F = \mathbb{Q}(t + \frac{2}{7})$.
   (c) Find the minimal polynomial of $t$ over $F$.

6. What is the Galois group of $X^3 + 7X + 7$ over
   (a) $\mathbb{F}_2$ (the field of 2 elements),
   (b) $\mathbb{F}_3$ (the field of 3 elements),
   (c) $\mathbb{Q}$.

   State clearly any results you use.

7. Suppose $A$ and $B$ are finitely generated abelian groups and $A \oplus A \cong B \oplus B$. Show that $A \cong B$.

8. Find the characteristic polynomial, invariant factors, elementary divisors, rational canonical form, and Jordan canonical form of the matrix
\[
\begin{pmatrix}
-1 & 1 & 1 \\
-2 & 2 & 1 \\
-2 & 1 & 2
\end{pmatrix}
\]