PhD Qualifying Exam: Algebra

April 9, 2005

Answer any five of the following eight questions.
You should state clearly any general results you use.

1. Suppose $G$ is a finite group and $K$ is a normal subgroup of $G$ with $\gcd(|K|, [G : K]) = 1$.
Show that $K$ is the unique subgroup of $G$ of order $|K|$.

2. Show that there are no simple groups of order 30.

3. Let $R$ be an integral domain. A non-zero non-unit element $s \in R$ is called special if for every $a \in R$ there exist $q, r \in R$ with $a = qs + r$ and such that either $r = 0$ or $r$ is a unit in $R$.
   (a) If $s \in R$ is special, show that the principal ideal $(s)$ is maximal.
   (b) Show that every polynomial of degree 1 in $\mathbb{Q}[X]$ is special in $\mathbb{Q}[X]$.
   (c) Prove that there are no special elements in $\mathbb{Z}[X]$.

4. Let $F$ be a field and let $R = \{ \sum_{i=0}^{n} a_i X^i : n \in \mathbb{N}, a_1 = 0 \}$ be the subring of the polynomial ring $F[X]$ consisting of all polynomials with $X$-coefficient equal to 0.
   (a) Show that $X^2$ is irreducible but not prime in $R$.
   (b) Show that the ideal of $R$ consisting of all polynomials in $R$ with constant term 0 is not principal.

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5. Suppose $F \subseteq \mathbb{C}$ and $F/\mathbb{Q}$ is a finite Galois extension with $\text{Gal}(F/\mathbb{Q})$ abelian. Let $\alpha \in F$ and assume $|\alpha| = 1$ where $|\alpha|$ is the absolute value of $\alpha$ considered as an element of $\mathbb{C}$.

(a) Show that $F$ is closed under complex conjugation. [Hint: $F/\mathbb{Q}$ is normal.]

(b) If $m_\alpha(X) \in \mathbb{Q}[X]$ is the minimal polynomial of $\alpha$ over $\mathbb{Q}$ and $\beta$ is another root of $m_\alpha$, show that $|\beta| = 1$. [Hint: use (a) and $\text{Gal}(F/\mathbb{Q})$ abelian.]

(c) Writing $m_\alpha(X) = \sum_{i=0}^n a_i X^i$ show that $|a_i| \leq 2^n$.

(d) Deduce that $F$ contains only finitely many $\alpha$ with $|\alpha| = 1$ and $m_\alpha \in \mathbb{Z}[X]$, and each of these is a root of unity.

6. Find the Galois group of $X^4 - 5X^2 + 6$ over

(a) $\mathbb{F}_3$ (the field with 3 elements),

(b) $\mathbb{F}_5$ (the field with 5 elements),

(c) $\mathbb{Q}$.

7. Let $A$ be an $4 \times 4$ matrix with complex entries and suppose $A^3 = A^2$. List all the possible Jordan canonical forms for $A$, and in each case give both the minimal and characteristic polynomials of $A$.

8. Show that if $A$ is a finite abelian group and $A \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) = 0$ for all primes $p$, then $A = 0$. Does this result remain true if $A$ is infinite? Explain.