PhD Qualifying Exam: Algebra

September 11, 2010

Answer any five of the following eight questions. You should state clearly any general results you use.

1. Prove that the symmetric group $S_n$ is a maximal subgroup of $S_{n+1}$, i.e., if $S_n \leq H \leq S_{n+1}$ then either $H = S_n$ or $H = S_{n+1}$. (Here we regard $S_n$ as the subgroup of permutations in $S_{n+1}$ that fix the element $n+1$.)

2. Let $N$ be a normal subgroup of $G$ and let $C$ be a conjugacy class of $G$ that is contained in $N$. Prove that if $[G : N] = p$ is prime, then either $C$ is a conjugacy class of $N$ or $C$ is a union of $p$ distinct conjugacy classes of $N$.

3. Let $R$ be a commutative ring with 1.
   (a) Show that if $M$ is a maximal ideal of $R$ then $M$ is a prime ideal of $R$.
   (b) Give an example of a non-zero prime ideal in a ring $R$ that is not a maximal ideal.
   (c) Show that if $R$ is finite then every prime ideal of $R$ is a maximal ideal.

4. Let $p$ be a prime and let $R$ be the ring of all $2 \times 2$ matrices of the form

   \[
   \begin{pmatrix}
   a & b \\
   pb & a
   \end{pmatrix}
   \]

   where $a, b \in \mathbb{Z}$. Prove that $R$ is isomorphic to $\mathbb{Z}[\sqrt{p}]$.

5. Show that the extension $\mathbb{Q}(\sqrt{2 + \sqrt{2}})/\mathbb{Q}$ is Galois, and describe its Galois group.

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6. An extension $K$ of a field $F$ of characteristic $p \neq 0$, is called purely inseparable extension if for each $\alpha \in K$ there is an integer $t$ such that $\alpha^{p^t} \in F$. Show that every purely inseparable field extension is a normal extension.

7. Let $T: \mathbb{Z}^n \to \mathbb{Z}^n$ be a $\mathbb{Z}$-linear map whose matrix with respect to the standard basis of $\mathbb{Z}^n$ is $M$. If $\det M \neq 0$, show that $\mathbb{Z}^n/\text{Im}(T)$ is a finite group of order $|\det M|$, where $\text{Im}(T)$ is the image of the map $T$.

8. Let $R$ be a commutative ring with 1 and let $N$ and $M$ be two $R$-modules. Prove that $N \otimes_R M \cong M \otimes_R N$. 

Page 2