Answer any **five** of the following eight questions.

**You should state clearly any general results you use.**

1. (a) Show that if $G$ is a nonabelian finite group then $|Z(G)| \leq \frac{1}{4}|G|$.
   
   (b) Give an example of a finite group with $|Z(G)| = \frac{1}{4}|G|$.

2. Let $G$ be a finite group acting on a set $X$ of size $n$ and suppose for any $a_1, a_2, b_1, b_2 \in X$ with $a_1 \neq a_2$ and $b_1 \neq b_2$, there exists a $g \in G$ such that $g \cdot a_i = b_i$ for $i = 1, 2$. Show that $|G|$ is divisible by $n(n-1)$. [Hint: consider the action of $\text{Stab}_G(a)$ on $X \setminus \{a\}$.]

3. Let $R$ be a ring with 1, and $n$ a positive integer. If $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in $R$, prove that $M_n(I)$ is an ideal of $M_n(R)$ whenever $I$ is an ideal of $R$, and that every ideal of $M_n(R)$ is of this form.

4. (a) Let $R$ be a PID. Show that if $P_1$ and $P_2$ are prime ideals with $P_1 \nsubseteq P_2$ then $P_1 = (0)$.
   
   (b) Give an example of a commutative ring and prime ideals $P_1, P_2$, with $(0) \nsubseteq P_1 \nsubseteq P_2$.

5. (a) Find the minimal polynomial $m_\alpha$ over $\mathbb{Q}$ of $\alpha = \sqrt{2 + \sqrt{6}}$.
   
   (b) Determine the Galois group of the splitting field extension of $m_\alpha$ over $\mathbb{Q}$.

6. Suppose $K = F(\alpha)$ is a non-trivial Galois extension of $F$ and assume there exists an element $\sigma \in \text{Gal}(K/F)$ such that $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

7. Let $R$ be an ID and $M$ an $R$-module. Define the rank $\text{rk}(M)$ of $M$ to be the maximum size of a $R$-linearly independent subset of $M$. Prove that for $n \in \mathbb{N}$, $\text{rk}(R^n) = n$, where $R^n$ denotes a direct sum of $n$ copies of $R$.

8. Let $R$ be a subring of a commutative ring $S$ and consider $S$ as an $R$-module. If $S$ is isomorphic (as a module) to a direct sum of $n$ copies of $R$, show that $S$ is isomorphic (as a ring) to a subring of $M_n(R)$, the ring of $n \times n$ matrices with entries in $R$. 