ANALYSIS QUALIFYING EXAM, 
SEPTEMBER 13, 2008

Do all 5 problems. Good luck.

1. (a) State carefully and precisely the Fundamental Theorem of Calculus for the Lebesgue integral.

(b) Let \( f(x) = x^\theta \sin(1/x) \) for \( 0 \leq x < 1 \) and \( f(0) = 0 \). For which real values of \( \theta \) is \( f \) absolutely continuous on \([0,1]\)?

2. Let \((\Omega, \Sigma, m)\) be a measure space. For \( A_n \in \Sigma \) let

\[
\limsup A_n = \left\{ x \in \Omega : x \in A_n \text{ for infinitely many positive integers } n \right\}.
\]

(a) Show that \( \limsup A_n = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k \) and conclude \( \limsup A_n \in \Sigma \).

(b) Assuming \( \sum_{n \geq 1} m(A_n) < \infty \), prove that \( m(\limsup A_n) = 0 \).

3. For \( j = 1,2 \) let

\[ f_j(t) = \int_0^\infty e^{-xt} g_j(x) dx \]

where \( g_j \) is continuous on \([0,\infty)\) and

\[ |g_j(x)| \leq 100 e^{\sqrt{x}} \]

for all positive \( x \).

(a) Prove that \( f_1 \) is continuous on \((0,\infty)\).

(b) Prove that \( \lim_{t \to \infty} f_1(t) = 0 \).

(c) Give examples of \( g_1, g_2 \) so that \( \lim_{t \to 0} f_1(t) = 5, \lim_{t \to 0} f_2(t) = -\infty \).

4. Let \( A : \text{Dom}(A) \subset H \to H \) be a linear operator satisfying the condition

\[ < Ax, y > = < x, Ay > \]
for all $x, y$ in the domain $Dom(A)$; here $< \cdot, \cdot >$ is the inner product on a complex Hilbert space $H$. Call $\Phi_j$ an eigenvector of $A$ corresponding to the eigenvalue $b_j$ if $\Phi_j$ is a nonzero vector in $Dom(A)$ and $A\Phi_j = b_j \Phi_j$; here $b_j$ is a complex number. Suppose that $b_1$ and $b_2$ are two different eigenvalues.

(a) Show that $b_1$ is real.

(b) Show that the corresponding eigenvectors satisfy $< \Phi_1, \Phi_2 > = 0$.

5. Consider two measures $m_1, m_2$ on $[0, \infty)$ equipped with its Borel sets; here $m_1$ is Lebesgue measure and $m_2$ has density $e^{-x}$. That is, for every Borel set $E$ in $[0, \infty)$,

$$m_2(E) = \int_E e^{-x} \, dx.$$ 

Let $M_j$ be the measure space $([0, \infty), \text{Borel sets}, m_j)$. Is there any containment relationship between $L^1(M_j)$ and $L^2(M_j)$? That is, either prove that

$$L^1(M_1) \subset L^2(M_1) \quad \text{or} \quad L^2(M_1) \subset L^1(M_1)$$

or give examples of functions in $L^2(M_1) \setminus L^1(M_1)$ and $L^1(M_1) \setminus L^2(M_1)$, and do the same thing with $m_2$ replacing $m_1$. 