Solve any 5 of the following 8 problems. Please write carefully and give sufficient explanations.

**Problem 1**

Let \( A_n \subset \mathbb{R}, \ n \in \mathbb{N} \). Define
\[
\lim A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n, \quad \liminf A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.
\]
Let \( m \) be the Lebesgue measure on \( \mathbb{R} \) and \( m^* \) its outer measure.

(a) Show that for any sequence of Lebesgue measurable sets \( A_n \) it holds
\[
m(\lim A_n) \leq \lim m(A_n).
\]
(b) Show that for any sequence of sets \( A_n \subset \mathbb{R} \),
\[
m^*(\lim A_n) \leq \lim m^*(A_n).
\]

**Problem 2**

Let \( f_n : A \rightarrow \mathbb{R}, \ A \in \mathcal{M}, \) be measurable and \( f_n \geq 0 \). Show:

(a) If \( \lim_{n \to \infty} \int_A f_n = 0 \) then \( f_n \to 0 \) in measure on \( A \).

(b) Give an example that the measure convergence cannot be replaced by convergence a.e.

**Problem 3**

Prove or disprove:
\[
\lim_{n \to \infty} \int_0^1 n^2 x(1-x)^n \, dx = \int_0^1 \lim_{n \to \infty} n^2 x(1-x)^n \, dx.
\]

**Problem 4**

Let \( f_n : \mathbb{R} \rightarrow \mathbb{R} \) be nonnegative and Lebesgue integrable functions on \( \mathbb{R} \) such that \( f_n \) is convergent to \( f \) on \( \mathbb{R} \). Assume also that \( \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) \, dx = \int_{\mathbb{R}} f(x) \, dx < \infty \). Show that for each Lebesgue measurable set \( A \subset \mathbb{R} \),
\[
\lim_{n \to \infty} \int_A f_n(x) \, dx = \int_A f(x) \, dx.
\]
HINT: Apply the Fatou Lemma.
Problem 5
State and prove the Minkowski Inequality in $L^p$ for $1 \leq p < \infty$.

Problem 6
(I) State the Radon-Nikodym Theorem for $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$.

(II) Let $\mu, \nu, \nu_i$, $i = 1, 2$, be $\sigma$-finite measures on the measurable space $(X, \mathcal{B})$. Let the symbol $\left[ \frac{d\nu}{d\mu} \right]$ denote the Radon-Nikodym derivative of $\nu$ with respect to $\mu$. Show:

(i) If $\nu$ is absolutely continuous with respect to $\mu$, that is $\nu \ll \mu$, and $f$ is a nonnegative measurable function, then
$$\int f \, d\nu = \int f \left[ \frac{d\nu}{d\mu} \right] \, d\nu.$$

(ii) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then
$$\left[ \frac{d(\nu_1 + \nu_2)}{d\mu} \right] = \left[ \frac{d\nu_1}{d\mu} \right] + \left[ \frac{d\nu_2}{d\mu} \right].$$

Problem 7
Recall that the space $\ell^\infty$ consists of all sequences $x = (\xi_j)$ such that $\|x\|_\infty = \sup_{j \in \mathbb{N}} |\xi_j| < \infty$.

(a) Show that $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \frac{\xi_j}{j}$ for $x = (\xi_j)$, is linear and bounded.

(b) Let $\mathcal{R}(T)$ be the range of $T$. Show that $\mathcal{R}(T)$ is not a closed subspace of $\ell^\infty$.

(c) Consider the inverse operator $T^{-1} : \mathcal{R}(T) \to \ell^\infty$, $\mathcal{R}(T) \subset \ell^\infty$. Show that $T^{-1}$ is unbounded.

Problem 8
Let $E = [0, 1] \times [0, 1]$, $m^2$ be the product Lebesgue measure on $\mathbb{R}^2$, and
$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$ Investigate the existence and equality of
$$\int_E f \, dm^2, \quad \int_0^1 \int_0^1 f(x, y) \, dx \, dy \quad \text{and} \quad \int_0^1 \int_0^1 f(x, y) \, dy \, dx.$$