1. Let $f : [a, b] \to \mathbb{R}$.
   (a) What does it mean for $f$ to be absolutely continuous?
   (b) Let $f$ be absolutely continuous and satisfy $f(x) > \varepsilon > 0$ for all $x \in [a, b]$. Prove that $g = 1/f$ is absolutely continuous.

2. State Hölder’s inequality for functions on the unit interval $[0,1]$. Prove that if $1 \leq p < r < \infty$, then $f \in L^r[0,1]$ implies $f \in L^p[0,1]$.

3. Let $(\Omega, \Sigma, \mu)$ be a measure space.
   (a) State Fatou’s Lemma for a sequence of $\Sigma$–measurable functions.
   (b) Use Fatou’s Lemma to prove Lebesgue’s Dominated Convergence Theorem, which you should state precisely.

4. Let $X$ be a Hilbert space. Let $\{f_n\}$ be a sequence in $X$.
   (a) Suppose $f_n$ converges weakly to $f$. Prove that $\|f_n\|$ is bounded.
   (b) Suppose that in (a), we assume in addition that $\|f_n\| \to \|f\|$. Then prove that $\|f_n - f\| \rightarrow 0$.

5. Prove or disprove the following statement. If $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is a sequence of Lebesgue integrable functions and $f_n \rightarrow 0$ in measure, then $f_n \rightarrow 0$ in $L^1(\mathbb{R})$.

6. Let $E \subset [0,1]$ have positive Lebesgue outer measure, and let $0 < a < 1$ be given. Prove that there is an interval $L$ such that the Lebesgue outer measure of $E \setminus L$ is at least $a$ times the length of $L$.

7. Show that any normed vector space can be isometrically embedded into a Banach space.

8. Let $f \in L^1(0,\infty)$. Define
   $$g(t) = \int_0^\infty e^{-tx}f(x)dx.$$
Prove that \( g \) is bounded and continuous on \([0, \infty)\) and

\[
\lim_{t \to \infty} g(t) = 0.
\]