Real Analysis Qualifying Exam
August, 2016

Please solve 4 of the following 7 problems, employing clarity of exposition and logical methodology.

(1) Let \((X, \mathcal{B}, \mu)\) be a probability space (i.e. \(\mu\) is a positive measure on the sigma-algebra \(\mathcal{B}\) with \(\mu(X) = 1\)). Recall that a sequence of real-valued measurable functions \(\{f_n\}_{n \in \mathbb{N}}\) is said to converge in measure to a measurable function \(f\) iff for each \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) so that for every \(n > N\),
\[
\mu\{x : |f_n(x) - f(x)| > \epsilon\} < \epsilon.
\]
(a) Show that if \(f_n \to f\) \(\mu\)-almost everywhere, then \(f_n \to f\) in measure. Give an example to show that the converse is false.
(b) Suppose each \(f_n \in L^1(\mu)\) and \(f_n \to f\) in \(L^1\). Show that \(f_n \to f\) in measure. Is the converse true? Prove or give a counterexample.

(2) Let \((X, \mathcal{B}, \mu)\) be a probability space (i.e. \(\mu\) is a positive measure on the sigma-algebra \(\mathcal{B}\) with \(\mu(X) = 1\)).
(a) Define the essential supremum of a measurable function \(f : X \to \mathbb{R}\).
(b) Define the space \(L^\infty(X, \mathcal{B}, \mu)\), and describe the usual norm.
(c) Show that \(f_n \to f\) in \(L^\infty(X, \mathcal{B}, \mu)\) iff there exists a set \(E \subset X\) with \(\mu(E) = 0\) so that \(f_n \to f\) uniformly off of \(E\).

(3) Recall that a Banach space is separable if there exists a countable, dense (in the norm topology) subset. Let \(\ell^1\) be the usual space of all absolutely summable real sequences, and \(\ell^\infty\) the space of bounded real sequences, each equipped with their usual norms.
(a) Show directly that \(\ell^1\) is separable.
(b) Show directly that \(\ell^\infty\) is not separable.

(4) Let \(I = [0, 1]\) denote the unit interval in \(\mathbb{R}\).
(a) Define the Cantor function \(f : I \to I\).
(b) Show that \(f\) is continuous and of bounded variation, but not absolutely continuous.

(5) Let \(X\) be a Banach space with \(\mathcal{L}(X)\) the space of bounded operators on \(X\), and let \(I \in \mathcal{L}(X)\) denote the identity operator \((I(x) = x \forall x \in X)\). Show the following:
(a) If \(T \in \mathcal{L}(X)\) and \(\|I - T\| < 1\) then \(T\) is invertible. Hint: consider an appropriate series.
(b) If \(T \in \mathcal{L}(X)\) is invertible and \(\|S - T\| < \|T^{-1}\|^{-1}\) then \(S\) is invertible.
Conclude that the set of invertible operators on \(X\) is open.
(6) Suppose that $f \geq 0$ is a (Lebesgue) measurable function on $\mathbb{R}$. Let 
$E_k = \{x : f(x) > 2^k\}$, and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$.
Show that

$$f \text{ is integrable iff } \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \text{ iff } \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty.$$ 

(7) Let $(X, \| \cdot \|)$ be a (real) Banach space and $\tau : X \to \mathbb{R}$ a linear functional on $X$. Show that $\tau$ is bounded if and only if the kernel of $\tau$,

$$\ker \tau = \{x \in X : \tau(x) = 0\},$$

is closed.