Linkedness, Orderedness and Connectivity in Graphs

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A graph $G = (V, E)$ is connected if for every $x, y \in V$ there is a path from $x$ to $y$. 
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A graph $G = (V, E)$ is $k$-connected if for every $x_1, \ldots, x_{k-1} \in V$ the graph $G - x_1 - \cdots - x_{k-1}$ is connected.
Definition

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Note

Similar concepts: edge-connectivity, connectivity and edge-connectivity for directed graphs.
Connectivity of Graphs

Note

$k$-connected $\neq$ $k$-edge-connected!
Connectivity of Graphs

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**Menger’s Theorem**

$G = (V, E)$ finite, undirected, $x, y \in V$, $\overline{xy} \notin E \Rightarrow$ minimum vertex cut for $x$ and $y = \text{the maximum number of pairwise vertex-independent paths from } x \text{ to } y.$
Connectivity of Graphs

Menger’s Theorem

\[ G = (V, E) \text{ finite, undirected, } x, y \in V, \ xy \notin E \Rightarrow \text{minimum vertex cut for} \ x \ \text{and} \ y = \text{the maximum number of pairwise vertex-independent paths from} \ x \ \text{to} \ y. \]

Note

Menger’s Theorem "Mutatis Mutandis" applies for directed graphs and for edge-connectivity as well.
Theorem

\[ G = (V, E) \text{ is } k\text{-connected} \Rightarrow \text{for } x, y \in V \text{ there exist } P_1, \ldots, P_k \text{ pairwise disjoint } x - y \text{ paths.} \]
Connectivity of Graphs

Theorem

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Proof:

Direct consequence of Menger’s Theorem.
Connectivity of Graphs

Theorem

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Proof:

Direct consequence of Menger’s Theorem.

Note

This property is in fact equivalent to \( k \)-connectivity.
Theorem

$G = (V, E)$ is $k$-connected $\Rightarrow$ for $x, y_1, \ldots, y_k \in V$ there exist $P_1, \ldots, P_k$ pairwise disjoint $x - y_i$ paths.
Theorem

\[ G = (V, E) \text{ is } k\text{-connected} \implies \text{for } x, y_1, \ldots, y_k \in V \text{ there exist } P_1, \ldots, P_k \]

pairwise disjoint \(x - y_i\) paths.

Theorem

\[ G = (V, E) \text{ is } k\text{-connected} \implies \text{for } x_1, \ldots, x_k \in V \text{ and } y_1, \ldots, y_k \in V \text{ there exist } P_1, \ldots, P_k \text{ pairwise disjoint } x_i - y_{\pi(i)} \text{ paths for some } \pi \in S_k. \]
Theorem

\( G = (V, E) \) is \( k \)-connected \( \Rightarrow \) for \( x, y_1, \ldots, y_k \in V \) there exist \( P_1, \ldots, P_k \) pairwise disjoint \( x - y_i \) paths.

Theorem

\( G = (V, E) \) is \( k \)-connected \( \Rightarrow \) for \( x_1, \ldots, x_k \in V \) and \( y_1, \ldots, y_k \in V \) there exist \( P_1, \ldots, P_k \) pairwise disjoint \( x_i - y_{\pi(i)} \) paths for some \( \pi \in S_k \).

Lemma

\( G = (V, E) \) is \( k \)-connected, \( H = (V', E') \) where \( V' = V \cup \{z\} \), 
\( E' = E \cup \{zx_i \mid x_i \in V, i = 1, \ldots, k\} \) \( \Rightarrow \) \( H \) is \( k \)-connected.
Theorem

Let $G = (V, E)$ be a graph. If $G$ is $k$-connected, then for any $x, y_1, \ldots, y_{k-1}, z \in V$, there exists a path $P$ from $x$ to $z$ containing $y_i$ for $i = 1, \ldots, k-1$ (in some order).

Theorem

Let $G = (V, E)$ be a graph. If $G$ is $k$-connected, then for any $x_1, \ldots, x_k \in V$, there exists a cycle $C$ containing $x_i$ for $i = 1, \ldots, k$ (in some order).
Quick Summary

We do not have control of the matching order or cyclic order of the given nodes.
**Quick Summary**

**Downside**

*We do not have control of the matching order cyclic order of the given nodes.*
Connectivity is insufficient (?) to gain control
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Question

How can we keep control?
Definition

\( G = (V, E) \) is \( k \)-linked if for \( x_1, \ldots, x_k \in V \) and \( y_1, \ldots, y_k \in V \) there exist \( P_1, \ldots, P_k \) pairwise disjoint \( x_i - y_i \) paths.
Definition

$G = (V, E)$ is $k$-linked if for $x_1, \ldots, x_k \in V$ and $y_1, \ldots, y_k \in V$ there exist $P_1, \ldots, P_k$ pairwise disjoint $x_i - y_i$ paths.

Definition

$G = (V, E)$ is $k$-routed if for $x_1, \ldots, x_k \in V$ there exist a path $P$ containing all $x_i$’s in the given order.
Definition

\( G = (V, E) \) is \( k \)-linked if for \( x_1, \ldots, x_k \in V \) and \( y_1, \ldots, y_k \in V \) there exist \( P_1, \ldots, P_k \) pairwise disjoint \( x_i - y_i \) paths.

Definition

\( G = (V, E) \) is \( k \)-routed if for \( x_1, \ldots, x_k \in V \) there exist a path \( P \) containing all \( x_i \)'s in the given order.

Definition

\( G = (V, E) \) is \( k \)-ordered if for \( x_1, \ldots, x_k \in V \) there exist a cycle \( C \) containing all \( x_i \)'s in the given cyclic order.
Example

The Cube \((K_2 \times K_2 \times K_2)\) is

- not 2-linked,
- 3-routed but not 4 routed,
- 3-ordered but not 4-ordered.
Proposition

- \( G = (V, E) \) \( k \)-linked \( \Rightarrow \) \( k \)-connected,
- \( G = (V, E) \) \( k \)-routed \( \Rightarrow \) \( k - 1 \)-connected,
- \( G = (V, E) \) \( k \)-ordered \( \Rightarrow \) \( k - 1 \)-connected.

Question

Does that work the other way around?
Linkage and Connectivity

Theorem

[Bollobas, Thomason, 1997] $G$ is $22k$ connected $\Rightarrow$ $G$ is $k$-linked.
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Theorem
[Thomas, Wollan, 2005] $G$ is $10k$ connected $\Rightarrow G$ is $k$-linked.
Summary

For $k \geq 3$,

- $2k$-connected $\not\iff$ $k$-linked $\Rightarrow$ $2k - 1$ - connected,
- $k$ - connected $\not\iff$ $k$-ordered $\Rightarrow$ $k - 1$ - connected,
- $k + 1$ - ordered $\not\iff$ $k$-linked $\Rightarrow$ $k$ - ordered,
- $k/2 + 1$ - linked $\not\iff$ $k$-ordered $\Rightarrow$ $k/2$ - linked,
- $k$ - linked $\not\iff$ $3k - 3$-connected, $10k$ connected $\Rightarrow$ $k$ - linked,
- $k$ - ordered $\not\iff$ $2k - 4$-connected, $10k$ connected $\Rightarrow$ $k$ - ordered,
Routedness

Proposition

- $G$ is $k$-ordered $\Rightarrow G$ is $k$-routed,
- $G$ is $k + 1$-routed $\Rightarrow G$ is $k$-ordered.

Note

$G$ is $k$-routed $\nRightarrow G$ is $k$-ordered.

Question

For which graphs does $k$-routedness imply $k$-orderedness?
Theorem

[Thomas, Wollan, 2005] \( G \) is \( 2k \) connected, \( e(G) > 5 \cdot n \cdot k \) \( \Rightarrow \) \( G \) is \( k \)-linked.
[Ng, Schultz] If $d(x) + d(y) \geq n + 2k - 6$ then $G$ is $k$-ordered Hamiltonian.
Theorem

[Ng, Schultz] If \( d(x) + d(y) \geq n + 2k - 6 \) then \( G \) is \( k \)-ordered Hamiltonian.

Recall:

Ore’s Theorem: \( d(x) + d(y) \geq n \) then \( G \) is Hamiltonian.
Theorem

[Godhad, 2002]

1. $G$ is 4-ordered planar graph $\Rightarrow$ $G$ is locally 3-connected maximal planar graph.

2. $G$ is 4-connected maximal planar graph $\Rightarrow$ $G$ is 4-ordered.

Theorem

[K. Meszaros, 2003] $G$ is 3-connected, 3-regular, 4-ordered $\Rightarrow g(G) \geq 5.$
Conjecture

$G$ is 6-connected $\Rightarrow$ $G$ is 4-ordered.
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$G$ is 6-connected $\Rightarrow$ $G$ is 4-ordered.

Conjecture

The $n$-dimensional hypercube ($K_2^n$) is $n$-ordered for $n \geq 3$. 
THANK YOU FOR YOUR ATTENTION!