On the maximum degree of path-pairable graphs

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Path-Pairable Graphs

Definition
For $k$ fixed, an undirected graph $G$ is $k$-path-pairable if, every ordered set of $2k$ pairwise disjoint vertices $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ there exist edge-disjoint paths $P_1, \ldots, P_k$ such that each $P_i$ is an $s_i; t_i$-path.

Definition
A graph $G$ on $2k$ vertices is path-pairable if it is $k$-path-pairable.
Asymptotic Behavior, Scalability

Observations

- Unlike many related parameters (linkedness, weak-linkedness), path-pairability does not require high edge density.
- On the other hand, path-pairable graphs all seem to have reasonable high maximum degree.
The Maximum Degree Problem

Theorem (Faudree, Gyárfás, Lehel, 1991)
There exist $k$-path-pairable graphs with maximum degree $\Delta = 3$ for arbitrary high values of $k$.

Theorem (Faudree, Gyárfás, Lehel, 1992)
If $G$ is a path-pairable graph on $n$ vertices with maximum degree $\Delta$, then $n \leq 2\Delta^\Delta$.

Proof
- Let $d = \log_\Delta \frac{n}{2}$.
- We can choose a pairing of the vertices of distance $d$ or more (why?).
- We need $\frac{n}{2} \cdot \log_\Delta \frac{n}{2}$ edges to build the paths but only have $\frac{n}{2} \cdot \Delta$. 
\[ \frac{\log n}{\log \log n} < \Delta_{min} < \ldots \]
The Maximum Degree Problem - Constructions

\[ \frac{\log n}{\log \log n} < \Delta_{min} < \ldots \]

**Advances**

- Csaba, Faudree, Gyárfás, Lehel, Schelp (1991): \( \Delta_{min} \leq 3 \cdot \sqrt{n} \)
- Lehel, Kubicza, Kubicky (1999): \( \Delta_{min} \leq 2 \cdot \sqrt{n} \)
- M. (2013): \( \Delta_{min} \leq 2\sqrt{2} \cdot \sqrt{n} \)
- M. (2014): \( \Delta_{min} \leq \sqrt{n} \)
- Győri, Mezei, M. (2016): \( \Delta_{min} \leq 5.2 \cdot \log n \)
Conjecture (Csaba, Faudree, Gyárfás, Lehel, Schelp, 1991)
Hypercubes of odd dimension are path-pairable.

Conjecture (Lehel, Kubicza, Kubicky, 1999)
Sufficiently large three dimensional complete grid graphs are path-pairable.
Terminal-Pairability in Graphs

Definition

Given a simple undirected graph $G$ and an undirected multigraph $D$ with $V(D) = V(G)$ we say that $G$ can realize the edges $e_1, \ldots, e_{|E(D)|}$ of $D$ if there exist edge disjoint paths $P_1, \ldots, P_{|E(D)|}$ in $G$ joining the endpoints of $e_1, \ldots, e_{|E(D)|}$, respectively.

Definition

A graph $G$ is terminal-pairable with respect to a family $\mathcal{F}$ of demand graphs on $V(G)$ if every demand graph in $\mathcal{F}$ can be realized by $G$. 
Problem (Csaba, Faudree, Gyárfás, Lehel, Schelp, 1991))

Let $G = K_n$ and let $\mathcal{F}_t = \{D : \Delta(D) \leq t\}$. What is the maximum of $t$ for which $G$ is terminal-pairable w.r.t. $\mathcal{F}_t$?
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Theorem (Csaba, Faudree, Gyárfás, Lehel, Schelp, 1991)

\( K_n \) is terminal-pairable w.r.t. \( \mathcal{F}_t \) for \( t \leq n^8 \).

\( K_n \) is not terminal-pairable w.r.t. \( \mathcal{F}_t \) for \( t > n^2 \).

Theorem (Győri, Mezei, M., 2016)

\( K_n \) is terminal-pairable w.r.t. \( \mathcal{F}_t \) for \( t \leq n^3 - c \).

Theorem (Girão, M., 2016)

\( K_n \) is not terminal-pairable w.r.t. \( \mathcal{F}_t \) for \( t > \frac{13}{27} n + c \).
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Theorem (Győri, Mezei, M.)

Let $G = K_n^t$ and let $D = (V(D), E(D))$ be a demand graph with $V(D) = V(K_n^t)$ and $\Delta(D) \leq \lfloor \frac{t}{6} \rfloor - 2$ even. Then every demand edge of $D$ can be assigned a path in $G$ joining the same endpoints such that the system of paths is edge-disjoint.

Corollary

If $t \geq 24$, $K_n^t$ is path-pairable.

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$\Delta_{\text{min}} \leq 2 \cdot \log n$. 

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If $t \geq 24$, $K_t^n$ is path-pairable.

Corollary

$\Delta_{min} \leq 5.2 \cdot \log n.$
Open Problems

- $\frac{\log n}{\log \log n} < \Delta < c \cdot \log n$
- $Q_n$ is path-pairable.
- Sharp bound on the terminal-pairability of complete graphs.
- Path-pairable planar graphs.
- $\Delta$-forcing in $k$-path-pairable graphs.