The path-pairability number of products of stars

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Abstract

A graph $G$ is $k$-path-pairable, if for any set of $2k$ distinct vertices, $s_i, t_i$, $1 \leq i \leq k$, there exist pairwise edge-disjoint $s_i, t_i$-paths in $G$, for $1 \leq i \leq k$. The path-pairability number is the largest $k$ such that $G$ is $k$-path-pairable. Here we determine the path-pairability number of the Cartesian product of two stars.

Keywords: path-pairability, weak linkage, Cartesian product, star, communication network

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1. Introduction

A telecommunications network (such as a data- or telephone network) is a collection of terminal nodes, links, and intermediate nodes that are assembled to enable communication between the terminals. In typical applications pairs of communicating terminals are connected through transmission links. In terms of graph theory, terminals and intermediate nodes of a network are the vertices
of a graph, the links and the transmission links correspond to edges and paths in the graph. The graph theory model of telecommunications networks and the various practical connectedness requirements imposed on the real networks lead to the notion of various linkage or pairing properties of a graph. Here we focus on the ‘$k$-path-pairability’ of graphs, a concept introduced by Csaba et al. [1] and has been investigated since then by several authors [2,3,4,6,8].

For $k$ fixed, a simple graph $G$ is $k$-path-pairable, if for any list of $2k$ distinct vertices (called terminals), $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$, there exist pairwise edge-disjoint $s_i, t_i$-paths in $G$, for $1 \leq i \leq k$.

The concept of $k$-path pairability is a weaker variant of weak $k$-linkedness, a property close to edge connectivity, where on the list $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ vertices may repeat. Indeed, a weakly $k$-linked graph must be $k$-edge connected (think of the choice of $s_i = s$, $t_i = t$, for $1 \leq i \leq k$, where $s \neq t$); on the other hand $k$-edge connectivity of a graph is ‘nearly’ sufficient for weak $k$-linkedness (for instance, Huck proved in [5] that, for $k$ odd, a $(k+1)$-edge-connected graph is weakly $k$-linked). In contrast, Faudree et al. showed in [2] that not even high degree is necessary for $k$-path pairability: for every $k$, and sufficiently large $n$, there is a 3-regular and $k$-path pairable graph of order $n$.

Let $pp(G) = \max\{k \mid G$ is $k$-path-pairable$\}$ be the path-pairability number of a graph $G$. For instance, $pp(K_{2,2}) = 1$, $pp(K_{1,b}) = \lceil b/2 \rceil$, $pp(K_n) = \lfloor n/2 \rfloor$.

A $k$-path pairable graph of order $2k$ is simply called path-pairable. The most obvious path-pairable graphs are $K_n$, for $n$ even, and $K_{1,a}$, for odd $a$. It is not quite obvious, but easy exercises to verify that $K_{a,b}$ is path-pairable, for all $a, b \geq 3$ and $ab$ even. The Petersen-graph and the 3-cube $Q^{(3)}$ are path-pairable graphs as well. A simple parity argument shows that the 2t-dimensional hypercube, $Q^{(2t)}$, is not path-pairable. Csaba et al. made the following appealing conjecture in 1992 (see [1]) which is still open: the hypercube $Q^{(2t+1)}$ is path-pairable, for every integer $t \geq 1$.

To find path-pairable graphs with small maximum degree it suffices to consider the Cartesian product of two cliques (such that the product has even order at least 6) or the Cartesian product of two complete $n \times n$ bipartite graphs (for $n$
even) (see [6,8]). It was proved recently by Győri et al. in [4] that the Cartesian $n$-th power of $K_t$ is path-pairable, for $t \geq 18$.

Here we discuss the path-pairability number of the Cartesian product of stars, the only possible path-pairable trees. For small values $t$ is straightforward to show $\text{pp}(K_{1,1} \square K_{1,1}) = \text{pp}(K_{1,1} \square K_{1,2}) = 1$, $\text{pp}(K_{1,2} \square K_{1,2}) = \text{pp}(K_{1,2} \square K_{1,3}) = 2$ (Proposition 12 in the Appendix). Our main result (proved in Section 2) answers a conjecture posed by Mészáros in [7].

**Theorem 1.** $\text{pp}(K_{1,a} \square K_{1,b}) = \lceil (a + b)/2 \rceil$, for every $a, b \geq 3$.

Theorem 1 shows that the path-pairability number of the Cartesian product of stars is not bounded. It is a natural question to ask whether the path-pairability number of the product of two non-star trees can be arbitrary high. For bounded path-paribility, consider the grid graph $P_n \square P_n$, where $P_n$ is a path on $n$ vertices. A cluster of six terminals in a $2 \times 3$ corner of the grid $P_n \square P_n$ is cut off from their six terminal pairs by $3 + 2 = 5$ edges, hence $\text{pp}(P_n \square P_n) \leq 5$ follows. In Section 3 a somewhat unexpected answer is obtained by using a strategy similar to the one in Theorem 1. If $\hat{K}_{1,m}$ denotes a star $K_{1,m}$ with a subdivided edge, then for $a, b \geq 3$, $\text{pp}(\hat{K}_{1,a} \square \hat{K}_{1,b}) \geq \lfloor \min\{a/2, b/2\} \rfloor$ (Proposition 8). Determining the exact value $\text{pp}(\hat{K}_{1,a} \square \hat{K}_{1,b})$ remains open.

## 2. The product of two stars

It was conjectured by Mészáros in [7] that $\text{pp}(K_{1,a} \square K_{1,b}) = \lceil (a + b)/2 \rceil$. We shall prove that the conjecture is true except particular parameter values discussed in the Appendix. Arrange the vertices of $G(a, b) = K_{1,a} \square K_{1,b}$ in an $(a + 1) \times (b + 1)$ matrix, where vertices are labeled $(i, j)$, $0 \leq i \leq a, 0 \leq j \leq b$, such that vertex $(0,0)$ has degree $a + b$, each vertex of the form $(0, j), j > 0$, has degree $a + 1$, each vertex of the form $(i, 0), i > 0$, has degree $b + 1$, and all other vertices have degree 2. Each row $A(i) = \{(i,j) \mid 0 \leq j \leq b\}$ induces a copy of $K_{1,b}$ with center $(i, 0)$, and each column $B(j) = \{(i,j) \mid 0 \leq i \leq a\}$ induces a copy of $K_{1,a}$ with center $(0, j)$. The set $\Gamma = A(0) \cup B(0)$ induces a star $K_{1,a+b}$ with center at $(0, 0)$, it is called the *boundary* star of $G(a,b)$.
Proposition 2. For $2 \leq a \leq b$ and $a + b \geq 6$, $pp(K_1, a \square K_1, b) \leq \lceil (a + b)/2 \rceil$.

Proof. For $a + b$ even, let $k = a + b + 2$, and define the pairs $s_\ell, t_\ell$, $1 \leq \ell \leq k/2$ as follows:

\begin{center}
\begin{tabular}{cccccccc}
  & $t_1$ & & & & & & \\
$s_1$ & $s_2$ & $s_3$ & $s_4$ & \cdots & $s_{k/2}$ & $t_{k/2}$ & $t_{k/2-1}$ & \cdots & $t_{a+2}$ \\
t_3 & $t_2$ & & & & & & \\
t_4 & & & & & & & & \\
\cdots & & & & & & & & \\
t_a & & & & & & & & \\
t_{a+1} & & & & & & & & \\
\end{tabular}
\end{center}

For $a + b$ odd, let $k = a + b + 3$, and define the pairs similarly:

\begin{center}
\begin{tabular}{cccccccc}
  & $t_1$ & & & & & & \\
$s_1$ & $s_2$ & $s_3$ & $s_4$ & \cdots & $s_{k/2}$ & $t_{k/2}$ & $t_{k/2-1}$ & \cdots & $t_{a+3}$ \\
t_4 & $t_2$ & $t_3$ & & & & & & \\
t_5 & & & & & & & & \\
\cdots & & & & & & & & \\
t_{a+1} & & & & & & & & \\
t_{a+2} & & & & & & & & \\
\end{tabular}
\end{center}

Observe that $k/2 \leq b + 1$, hence there is room for $s_{k/2}$ in $A(1)$. The $k/2$ pairwise edge disjoint $s_i, t_i$-paths, $1 \leq i \leq k/2$, if exist, called a solution. In each case, either $s_2s_1$ or $s_2t_1$ is the first edge of the $s_2, t_2$-path $P_2$ of a solution. Since all the vertices of degree two in $A(1) \cup B(1)$ are occupied by a terminal, $P_2$ must proceed using either the edge $s_1 - (0, 0)$ or the edge $t_1 - (0, 0)$. On the other hand, for the same reason, the $s_1, t_1$-path $P_1$ must use both edges $s_1 - (0, 0)$ and $t_1 - (0, 0)$, Thus $P_1$ and $P_2$ cannot be edge-disjoint. $\square$
2.1. Mating terminals

To verify that $G(a, b) = K_{1,a} \square K_{1,b}$ is $k$-path pairable, we must show that given any $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of $G(a, b)$, we can find $k$ pairwise edge-disjoint $s_\ell, t_\ell$-paths, for $1 \leq \ell \leq k$. Our procedure to obtain these $s_\ell, t_\ell$-paths consists of two basic steps. First we assign distinct boundary vertices $\phi(s_\ell), \phi(t_\ell) \in \Gamma$, called mates of $s_\ell$ and $t_\ell$, such that they can be reached from $s_\ell$ and from $t_\ell$ via pairwise edge-disjoint mating paths. Second we join each pair of mates using paths of length two through the center vertex in $\Gamma$. The concatenation of these three pieces, for each $1 \leq \ell \leq k$, yields a solution, the required $k$ pairwise edge-disjoint paths.

For a terminal $u = (i, p) \notin \Gamma$, the vertices $(i, 0)$ and $(0, p)$ are the possible 1-step boundary mates of $u$ reached by the mating path $u - (i, 0)$ or $u - (0, p)$. The vertices $(0, q), p \neq q$, and $(j, 0), i \neq j$, are the possible 3-step boundary mates of $u$. A 3-step boundary mate of $u$ is reached by the mating path $u - (i, 0) - (i, q) - (0, q)$ or $u - (0, p) - (j, p) - (j, 0)$ through transit vertex $(i, q)$ or $(j, p)$.

Lemma 3. For $n \leq a + b$, and for any family $X$ of $n$ distinct terminals of $G(a, b)$, there is an injection $\phi : X \longrightarrow \Gamma$ such that $\phi(x) = x$, for every $x \in X \cap \Gamma$, and $\phi(x)$ is a 1- or 3-step boundary mate of $x$, for every $x \in X \setminus \Gamma$. Furthermore, the mating paths determined by $\phi$ are pairwise edge-disjoint.

Proof. Let $L_0 = X \cap \Gamma$ and set $\phi(x) = x$, for each $x \in L_0$. The definition of $\phi$ is successively extended for the terminals in $X \setminus L_0$ via an algorithm that repeatedly updates a set $L_1 \cup L_3$ of terminals and a set $L'_1 \cup L'_3$ of associated mates. At any given stage of the algorithm $L_i$ contains all terminals $t$ associated with an $i$-step boundary mate $t' \in L'_i$, for $i = 1, 3$. An auxiliary set $T$ is also maintained and updated including the transit vertices belonging to the mating paths of length 3 between a terminal $t \in L_3$ and its actual mate $t' \in L'_3$. Set $L = L_0 \cup L_1 \cup L_3$, $L' = L_0 \cup L'_1 \cup L'_3$. The algorithm terminates when $L = X$, and then $\phi(t) = t'$, for all $t \in X$.

If $L \neq X$, there is an unmated terminal $t \in X \setminus L$, and since $|L| < |X| \leq a + b$,
there is an unused boundary vertex $y \in \Gamma \setminus L'$. Let $t = (i, p)$ and $y = (j, 0)$ (or $y = (0, q)$). The mate of $t$ is assigned as follows.

1. If $i = j$ (or $p = q$), then we add $t$ and $t' = y$ to $L_1$ and $L'_1$, respectively. Repeat this step until $L_1$ becomes maximal. Thus we assume that for any unmated terminal $(i, p) \in X \setminus L$, we have $(i, 0), (0, p) \in L'$, in particular, for every unused boundary vertex $y = (j, 0)$ (or $y = (0, q)$), we have $i \neq j$ (or $p \neq q$).

2. If $(j, p) \notin X \cup T$, then the path $t - (0, p) - (j, p) - y$ does not interfere with the mating paths used between the terminals in $L$ and their mates in $L'$. Now we redefine the sets $L_3, L'_3$, and $T$ by including $t$ into $L_3$, $(j, p)$ into $T$, and $t' = y$ into $L'_3$.

3. If $u = (j, p) \in X$, then by (1) above, we have $u \in L_1$ and $u' = (0, j) \in L'$. Then we redefine the 1-step mate of $u$ by letting $u' = y$ and assigning $t' = (0, j)$ as the the 1-step mate of $t$. The sets $L_1$ and $L'_1$ are updated corresponding to the changes in the mapping $\phi$. 

(1)  
\[
\begin{array}{cccc}
0 & p & \text{t} \\
0 & \text{i} & \text{y} & \text{t}
\end{array}
\]

(2)  
\[
\begin{array}{cccc}
0 & p & \text{t} \\
0 & \text{i} & \text{y} & \text{t}
\end{array}
\]

(3)  
\[
\begin{array}{cccc}
0 & p & \text{u} \\
0 & \text{i} & \text{y} & \text{u}
\end{array}
\]

(4)  
\[
\begin{array}{cccc}
0 & q & p & \text{x} \\
0 & \text{i} & \text{y} & \text{u}
\end{array}
\]
(4) If \( x = (j, p) \in T \) is a transit vertex, then it is used by a 3-path between some \( u = (j, q) \in L_3 \) and its mate \( u' = (0, p) \in L'_3 \). Now we redefine the 3-step mate of \( u \) by letting \( u' = y \) its new 1-step mate, and assigning \( t' = (0, p) \). Then \( t \) is included into \( L_1 \), \( u \) is transferred from \( L_3 \) into \( L_1 \), vertex \((j, p)\) is removed from \( T \), and \( L'_1, L'_3 \) are updated accordingly. □

2.2. Mating lemma extended

If there are \( a + b + 1 \) terminals, there is not enough room for the mates in \( \Gamma \setminus \{(0, 0)\} \) and the procedure in Section 2.1 must be modified. The strategy is simple, before injecting the terminals into \( \Gamma \) we ‘reserve’ an \( s_{\ell}, t_{\ell} \)-path of length at most 6, for some \( \ell \), such that this reserved path does not use edges in \( \Gamma \).

Reserving a path is done by ‘blocking’ the participating edges from being used in the mating procedure. Lemma 4 extends Lemma 3 just for this purpose.

**Lemma 4.** Let \( X \) be a family of \( a + b \) distinct terminals in \( G(a, b) \) and let \( Z \) be a set of at most three vertices of \( G(a, b) \setminus X \). Assume that for a set \( X_0 \subseteq X \) there is an injective pre-map \( \phi_0 : X_0 \rightarrow \Gamma \) such that:
- \( \phi_0(x) = x \), for every \( x \in X_0 \cap \Gamma \),
- \( \phi_0(x) \) is a 1- or 3-step boundary mate of \( x \), for every \( x \in X_0 \setminus \Gamma \),
- \( (0, j), (i, 0) \in \phi_0(X_0) \), for every \((i, j) \in Z \),
- the mating paths determined by \( \phi_0 \) are edge-disjoint and disjoint from \( Z \).

Then \( \phi_0 \) has an extension to an injection \( \phi : X \rightarrow \Gamma \) such that \( \phi(x) = x \), for \( x \in X \cap \Gamma \), \( \phi(x) \) is a 1- or 3-step boundary mate of \( x \), for every \( x \in X \setminus \Gamma \), and the mating paths determined by \( \phi \) are edge-disjoint and disjoint from \( Z \).

**Proof.** We use the notation of Lemma 3. In particular, \( L_0 = X \cap \Gamma \). Observe that the case \( Z = \emptyset \) with \( X_0 = L_0 \) and the identity map \( \phi_0 : X_0 \rightarrow X_0 \) is the claim we proved by using the algorithm in Lemma 3.

For \(|Z| \geq 1\), we repeat the algorithm in Lemma 3 by setting the initial values implied by the partial injection \( \phi_0 \) as follows. Let \( L = X_0 \) and \( L' = \phi(X_0) \) partitioned into \( L_0 \cup L_1 \cup L_3 \) and \( L_0 \cup L'_1 \cup L'_3 \), respectively, where \( L'_h \) is the set of all \( h \)-step mates assigned by \( \phi_0 \) to the terminals in \( L_h \), for \( h = 1, 3 \).
Furthermore, let \( T_0 \) be the set of all transit vertices used in the mating paths of length 3 between \( x \in X_0 \) and its mate \( \phi_0(x) \), and set \( T = T_0 \cup Z \).

If \( L \neq X \), there is an unmated terminal \( t = (i, j) \in X \setminus L \), and there is an unused boundary vertex \( y = (p, 0) \in \Gamma \setminus L' \) (or \( y = (0, q) \in \Gamma \setminus L' \)). Observe that \( (p, j) \notin Z \) (or \( (i, q) \notin Z \)), since otherwise, \( (p, 0) \in L' \) (or \( (0, q) \in L' \)), by assumption. As a consequence, a terminal in \( Z \) is never considered in cases (1)–(4) of the algorithm in Lemma 3. In fact, \( t, u, x, y \notin Z \) in the figures above. Thus Lemma 3 yields the required extension \( \phi \). \( \Box \)

When we apply the mating lemma an \( s_t, t_\ell \)-path will be reserved by specifying a set \( Z \) of at most 3 vertices containing \( t_\ell \), together with \( X_0 \) and \( \phi_0(X_0) \) such that the assumptions of Lemma 4 are satisfied. We will say that the mates in \( \phi_0(X_0) \) are ‘blocking’ the terminal and the transit vertices in \( Z \).

2.3. The row-column bipartite graph

Let \( H \) be the \( a \times b \) bipartite graph of the rows and columns of \( G(a, b) \) with vertex set \( V(H) = \{A(1), \ldots, A(a)\} \cup \{B(1), \ldots, B(b)\} \), an there is an edge between vertices \( A(i) \) and \( B(j) \), \( 1 \leq i \leq a, 1 \leq j \leq b \), if and only if \( (i, j) \in X \). (Here we assume that \( A(0) \cup B(0) \) contains no terminal; this restriction will be reconsidered later.) We will need an elementary lemma in Section 2.4.

![Figure 1: The base graphs in Lemma 5](image)

**Lemma 5.** Let \( \mathcal{H} \) be the family of all connected bipartite graphs such that each \( H \in \mathcal{H} \) has \( |V(H)|+1 \) edges and the degree of each vertex in one of the partition classes of \( H \) is equal to 2. Then every member of \( \mathcal{H} \) can be obtained starting...
with one of the base graphs \( \Lambda, \Theta, \Sigma \) in Fig. 1 and by repeating even extensions: subdividing an edge with an even number of vertices or suspending a path of even length at any vertex of the smaller partition class.

**Proof.** Let \( H \in \mathcal{H} \) be a bipartite graph with partition classes \( |A| = a \) and \( |B| = b \), and assume that the vertices of \( B \) have degree 2. Because \( H \) has \( a + b + 1 = 2b \) edges we obtain \( b = a + 1 \).

First observe that starting with \( H \) any of the described even extensions produce members in \( \mathcal{H} \).

Assume now that \( H \in \mathcal{H} \) is minimal with respect to the two even extensions, in particular, \( d_H(x) \geq 2 \), for every \( x \in A \), and

for any 4-path \((x_0, y, x, y_0)\), if \( x_0 \in A \) and \( d_H(x) = 2 \), then \( x_0y_0 \in E(H) \). (*)

We show that \( H \) is one of the graphs \( \Lambda, \Theta, \Sigma \). Since \( H \) has \( 2a + 2 \) edges incident with a vertex of degree at least 2, the possible degree sequence of the vertices in \( A \) are

\[
(4, \underbrace{2, \ldots, 2}_{a-1}) \quad \text{and} \quad (3, \underbrace{2, \ldots, 2}_{a-2}).
\]

If \( x \in A \) has degree 4, then by (*), \( H - x \) is the union of two paths of length 2, and \( H \cong \Lambda \) follows. If \( x_1, x_2 \in A \) are the two vertices of degree 3, then connectivity of \( H \) and (*) imply that \( x_1, x_2 \) have a common neighbor \( y \). Then \( H - \{x_1, x_2, y\} \) induces two paths of length 0 or 2. Using property (*) again, we obtain \( H \cong \Theta \) or \( H \cong \Sigma \). \( \Box \)

Observe that with the only exception of \( \Theta \), every graph in \( \mathcal{H} \) has a path of length 4 with end vertices lying in the smaller partition class. Lemma 5 has no natural extension for non-connected graphs, we will use the following observation instead.

**Proposition 6.** If a non-connected bipartite graph \( H \) contains \( |V(H)| + 1 \) edges and the degree of each vertex in one of the partition classes is equal to 2, then
$H$ has a connected component which is either a cycle or an even path with end vertices in the smaller partition class.

Proof. Let $H$ be an $a \times b$ bipartite graph, $a < b$, and let $A$ and $B$ be the partition classes with $|A| = a$, $|B| = b$. Let $H_i$, $1 \leq i \leq c$, be the connected components of $H$, where each $H_i$ is an $a_i \times b_i$ bipartite graph, $\sum a_i = a$ and $\sum b_i = b$. By assumption, each vertex in $B$ has degree 2, and $H$ has $2b = a + b + 1$ edges.

By connectivity, we have $2b_i \geq a_i + b_i - 1$, for every $i = 1, \ldots, c$, with equality if $H_i$ is an even path with end vertices in $A$. If it does not happen, then $2b_i \geq a_i + b_i$, for every $i = 1, \ldots, c$, with equality provided $H_i$ is an even cycle. Assuming that this is not the case either, we have $2b_i \geq a_i + b_i + 1$, for every $i = 1, \ldots, c$. Adding up these inequalities results in $a + b + 1 = 2b \geq a + b + c$. Thus we obtain $c = 1$, a contradiction. $\square$

2.4. Proof of the main theorem

Theorem 7. $pp(K_1,a \Box K_1,b) = \lceil (a + b)/2 \rceil$, for $2 \leq a \leq b$, except the case $a = 2, b = 3$.

Proof. Let $k = \lceil (a + b)/2 \rceil$. By Proposition 2, it is enough to show that $G(a, b) = K_{1,a} \Box K_{1,b}$ is $k$-path pairable. Let $X = \{s_1, t_1, \ldots, s_k, t_k\}$ be any set of $2k$ distinct terminals.

For $a + b$ even, $2k = a + b$, and by Lemma 3 it follows that the terminals have an injection into 1- or 3-step mates on the boundary $\Gamma$ of $G(a, b)$ along with edge-disjoint paths. For every pair $s_\ell, t_\ell$, $1 \leq \ell \leq k$, the mates $\phi(s_\ell)$ and $\phi(t_\ell)$ can be joined in $\Gamma$. The concatenation of these three pieces produces the required $s_\ell, t_\ell$-paths.

From now on we assume that $a + b$ is odd, in particular, $b \geq a + 1$ and $2k = a + b + 1$. In this case $\Gamma - (0, 0)$ cannot receive the mates of all terminals, one pair, $s_\ell, t_\ell$, must be joined with a path $P_\ell$ not using edges of $\Gamma$. To make sure that the remaining pairs have an injection into $\Gamma$ we need to ‘reserve’ $P_\ell$ by ‘blocking’ the edges of $P_\ell$ from being used in the mating procedure. In a case-by-case analysis we apply Lemma 4 that sets up a blocking by specifying an
appropriate pre-map $\phi_0$. The first two types of blocking are used when members of some pair of terminals lie in the same row or column.

(a) Let $s_\ell = (i, p) \in X, t_\ell = (j, p) \in X$, where $0 \leq i, j \leq a$ and $1 \leq p \leq b$. We reserve the path $s_\ell - (0, p) - t_\ell$ (or $s_\ell - t_\ell$). Let $u \in X \setminus B(p)$ be a terminal closest to $(j, 0)$, and apply Lemma 4 with $X \setminus \{t_\ell\}, Z = \{t_\ell\}, \phi_0(s_\ell) = (0, p)$, and $\phi_0(u) = (j, 0)$. (In the figure $u' = \phi_0(u)$ is a 1-step mate of $u$.)

(b) Let $s_\ell = (i, q) \in X, t_\ell = (i, p) \in X$, where $0 \leq p, q \leq b$ and $1 \leq i \leq a$. We reserve the path $s_\ell - (i, 0) - t_\ell$ (or $s_\ell - t_\ell$). Let $u \in X \setminus A(i)$ be a terminal closest to $(0, p)$. Now we apply Lemma 4 with $X \setminus \{t_\ell\}, Z = \{t_\ell\}, \phi_0(s_\ell) = (i, 0)$, and $\phi_0(u) = (0, p)$.

(Concerning the choice of a ‘closest’ terminal $u$ in the blockings above figure (a) illustrates the case $(X \cap A(j)) \setminus \{t_\ell\} \neq \emptyset$ when $u'$ is a 1-step mate of $u$; figure (b) illustrates the case $(X \cap B(p)) \setminus \{t_\ell\} = \emptyset$ when $u'$ is a 3-step mate of $u$.)

Due to (a) and (b), for every pair of terminals $s_\ell = (i, p), t_\ell = (j, p)$, we may assume that $i \neq j$ and $p \neq q$. We also assume $(0, 0) \notin X$, since otherwise, $\Gamma$ can be filled with all the $a + b + 1$ terminals using Lemma 3 and we are done.

First we are dealing with terminals in $X \cap \Gamma$. For each $x = (0, q) \in X$, if $(i, q) \notin X$, for some $1 \leq i \leq a$, then we replace $x$ with $x^* = (i, q)$. Suppose now that there is a solution with the modified set $X^*$ of terminals. The $s_\ell^*, t_\ell^*$-path $P$ either contains an $s_\ell, t_\ell$-path, or one (or both) of the edges $s_\ell^* - s_\ell$ and $t_\ell^* - t_\ell$ are not used in the pairing, in which case they can be added to $P$ forming an $s_\ell, t_\ell$-path. From a solution for the pairs in $X^*$ thus we derive a solution for the original set $X$ of terminals. The same reasoning can be repeated for the
terminals lying in $B(0)$. Thus we may assume that

$$(0, q) \in X \text{ implies } (i, q) \in X, \text{ for every } 1 \leq i \leq a,$$

$$(p, 0) \in X \text{ implies } (p, j) \in X, \text{ for every } 1 \leq j \leq b.$$  \hspace{1cm} (1)

Since $2(b+1) > a+b+1$, we have $|X \cap B(0)| \leq 1$, and in the case of equality, we also have $m = |X \cap A(0)| \leq 1$. Observe that for $X \cap B(0) = \emptyset$, the average number of terminals per column is $(a+b+1)/b \leq 2$ with equality for $b = a+1$.

**Case I.** Every column $B(j)$, $1 \leq j \leq b$, contains exactly two terminals.

Since $a \geq 2$, we have $X \cap \Gamma = \emptyset$ and $a = b - 1$. Let $H$ be the $a \times b$ raw-column bipartite graph of $G(a,b)$ where each edge $(i,j) \in H$ corresponds to the terminal $(i,j) \in X$ located at $A(i) \cap B(j)$.

Assume that $H$ is not connected. Then, by Proposition 6, $H$ has a component $H_1$ which is a cycle or an even path. Let $H_0 = H - H_1$. If there is a pair $s_\ell , t_\ell$ separated by $H_0$ and $H_1$, say $s_\ell \in H_0$, $t_\ell \in H_1$, then blocking a path between them will be done as follows.

$(\gamma)$ Let $s_\ell = (i,p) \in X$, $t_\ell = (j,q) \in X$, with $i \neq j$, $p \neq q$; then $(i,q) \notin X$ (since $s_\ell , t_\ell$ are in distinct components). We reserve the path $s_\ell - (i,0) - (i,q) - (0,q) - t_\ell$. Let $u \in (X \cap B(p)) \setminus \{s_\ell\}$ and let $v \in (X \cap B(q)) \setminus \{t_\ell\}$. We apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell, (i,q)\}$, $\phi_0(u) = (j,0)$, $\phi_0(v) = (0,q)$, and $\phi_0(s_\ell) = (i,0)$.

Notice that $(\gamma)$ can be used under the weaker condition that $(i,q) \notin X$ and each of $B(p)$ and $B(q)$ contains at least two terminals (if $|X \cap B(p)| > 2$ then $u$ is selected to be the closest terminal to $(j,0)$).
Now we assume that there is no pair of terminals separated by distinct connected components of $H$. Let $s_\ell, t_\ell$ belong to $H_1$. Assuming that $(\gamma)$ does not apply, and since $H_1$ is either an even path or a cycle, we obtain that $H_1 \cong K_{2,2}$ containing another pair, denote them simply $u, v$. Reserving a path for $(s_\ell, t_\ell)$ is done next.

$(\delta)$ Let $s_\ell = (i, p), t_\ell = (j, q), u = (i, q), v = (j, p)$ (with $i \neq j, p \neq q$), and let $w = (h, r) \in X$, for $h \notin \{i,j\}, r \notin \{p,q\}$. To reserve the path $s_\ell - (i, 0) - (i, r) - (0, r) - (j, r) - (j, 0) - t_\ell$ we apply Lemma 4 with $X \setminus \{t_\ell\}, Z = \{t_\ell, (i, r), (j, r)\}, \phi_0(s_\ell) = (i, 0), \phi_0(u) = (0, q), \phi_0(w) = (0, r)$, and $\phi_0(v) = (j, 0)$.

![Figure 2: $(\gamma)$ and $(\delta)$ in terms of $H$](image)

Next let $H$ be connected. By Lemma 4, $H$ is either isomorphic to a base graph $H_0 \in \{\Lambda, \Sigma, \Theta\}$ or obtained from a base graph by even extensions. Observe that $H \not\cong \Theta$, since otherwise we have the case $a = 2, b = 3$ excluded in the theorem.

A fairly straightforward case analysis shows that $H$ always contains an appropriate pair $s_\ell, t_\ell$, such that $s_\ell$ is an edge of $H$ belonging to an eventually subdivided copy of a $K_{2,2} \subset H_0$, together with $t_\ell, u, v$ and $w$, as shown on Fig 2 (dashed lines indicate non-edges of $H$). In the first two cases $(\gamma)$ works, in the third case $(\delta)$ applies.

**Case II.** There is a column not in $\Gamma$ which contains exactly one terminal. Let $X \cap B(b) = \{s_\ell\}, s_\ell = (i, b)$, and $t_\ell = (j, q)$. Observe that $i \neq 0$ and assume $i \neq j$. We reserve the path $s_\ell - (0, b) - (j, b) - (j, 0) - t_\ell$. Let $u \in X \setminus \{s_\ell, t_\ell\}$, be a terminal closest to $(j, 0)$. Let $v \in X \setminus \{s_\ell, t_\ell\}$, be a terminal different from $u$ and closest to $(0, q)$. (Such $u$ and $v$ exist, because $|X| \geq b + 3$.) Now we
apply Lemma \[4\] with \(X \setminus \{t_\ell\}, Z = \{t_\ell,(j,b)\}, \phi_0(s_\ell) = (0,b), \phi_0(u) = (j,0), \phi_0(v) = (0,q)\).

**Case III.** There is a column containing no terminal. Let \(X \cap B(b) = \emptyset\), and assume that Case II does not apply, hence each column has 0 or at least 2 terminals. Recall that \(|X \cap B(0)| \leq 1\), and either \(|X \cap A(0)| \leq 2\) or \(|X \cap \Gamma| = |X \cap A(0)| \geq 3\).

**III.1.** First we assume that \(X' = X \setminus \Gamma\) is not contained in the union of two rows. Let \(s_\ell = (i,p) \in X', t_\ell = (j,q) \in X'\) (with \(i \neq j, p \neq q\)). Since \(|X \cap B(0)| \leq 1\), w.l.o.g. we may assume that \((i,0) \notin X\). We reserve the path \(s_\ell - (i,0) - (i,b) - (0,b) - (j,b) - (j,0) - t_\ell\).

By assumption, there is a terminal \(u \in X' \setminus (A(i) \cup A(j))\); assume w.l.o.g that \(u \notin B(q)\). Next we select a terminal \(v_1 \in (X \cap B(q)) \setminus \{t_\ell\}\) closest to \((0,q)\). There still remain terminals not in \(B(q) \cup \{u,s_\ell\}\), let \(v_2 \in X \setminus (B(q) \cup \{u,s_\ell\})\) be a terminal closest to \((j,0)\). Now we apply Lemma \[4\] with \(X \setminus \{t_\ell\}, Z = \{t_\ell,(i,b),(j,b)\}, \phi_0(s_\ell) = (i,0), \phi_0(u) = (0,b), \phi_0(v_1) = (0,q), \phi_0(v_2) = (j,0)\).

**III.2.** Next we assume that \(m = |X \cap A(0)| \leq 2\), and \(X' = X \setminus \Gamma\) belongs to the union of two rows, say \(X' \subseteq A(i) \cup A(j)\) \((1 \leq i < j \leq a)\).

For \(s_\ell = (i,p) \in X, t_\ell = (j,q) \in X\) (with \(p \neq q\)), we reserve the vertices of the path \(s_\ell - (i,0) - (i,b) - (0,b) - (j,b) - (j,0) - t_\ell\) not in \(B(b)\). Then we remove the pair \(s_\ell, t_\ell\) from \(X\) and apply Lemma \[4\] on \(G(a,b-1) = G(a,b) - B(b)\) with the remaining \((a + b - 1)/2\) pairs. For \(r \notin \{p,q\}\), let \(B(r)\) be a column containing (two) terminals. Let \(Z = \{s_\ell, t_\ell\} \phi_0(j,p) = (0,p), \phi_0(i,q) = (0,q)\).
3. The product of two trees different from a star

Set $\phi_0(i, r) = (i, 0)$ and $\phi_0(j, r) = (j, 0)$, unless one of $(i, 0)$ or $(j, 0)$ is a terminal, in which case we set $\phi_0(i, 0) = (i, 0)$ or $\phi_0(j, 0) = (j, 0)$, respectively.

III.3. Finally we consider the case when $m = |X \cap A(0)| \geq 3$.

Since $m(a+1) \leq a+b+1$ and $a \geq 2$, we obtain $m < (a+b+1)/2$. Hence there is a pair of terminals not in $A(0)$, let $s_\ell = (i, p), t_\ell = (j, q)$, where $1 \leq i, j \leq a$ and $1 \leq p, q < b$ (with $i \neq j, p \neq q$). Since $m \geq 3$, there are distinct terminals $u_1, u_2 \in A(i) \setminus \{s_\ell\}$, and there are distinct terminals $v_1, v_2 \in A(j) \setminus \{t_\ell\}$ such that $u_2$ is closest to $(0, q)$ and $v_2$ is closest to $(0, p)$.

We reserve the path $s_\ell - (i, 0) - (i, b) - (0, b) - (j, b) - (j, 0) - t_\ell$ as in III.2, and remove an appropriate pair of terminals from $X$. Then we apply Lemma 4 on $G(a, b-1) = G(a, b) - B(b)$ with the set of terminals $X \setminus \{s_\ell, t_\ell\}$, and by setting $Z = \{s_\ell, t_\ell\}$, $\phi_0(u_1) = (i, 0), \phi_0(v_1) = (j, 0), \phi_0(u_2) = (0, q), \phi_0(v_2) = (0, p)$. \qed

3. The product of two trees different from a star

A $k$-path-pairable graph $G$ obviously satisfies the condition that for any subset $A \subset V(G)$, $|A| \leq k$, the number of edges form $A$ to $V(G) \setminus A$ is at least $k$. Based on this ‘cut condition’ it follows easily that the Cartesian product of a non-star tree $T$ with any path $P$ of length at least three has path-pairability number at most 5. Since $pp(T) = pp(P) = 1$, one would expect that for any trees $T_1, T_2$ such that $pp(T_1) = pp(T_2) = 1$, we have $pp(T_1 \square T_2) \leq c$, with some constant $c$. Our next result shows that this is not always the case.

**Proposition 8.** If $\hat{K}_{1,m}$ denote a star $K_{1,m}$ with a subdivided edge, then for $a, b \geq 3$, $pp(\hat{K}_{1,a} \square \hat{K}_{1,b}) \geq \lfloor \min\{a/2, b/2\} \rfloor$.

**Proof.** Let $x_0 \in V(\hat{K}_{1,a})$ and $y_0 \in V(\hat{K}_{1,b})$ be the leaf incident to the subdivided edge, and let $x_1$ and $y_1$ be its neighbor of degree two, respectively. We use the notations $A(x_0) = \{x_0\} \square \hat{K}_{1,b}$, $A(x_1) = \{x_1\} \square \hat{K}_{1,b}$, and $B(y_0) = \hat{K}_{1,a} \square \{y_0\}$, $B(y_1) = \hat{K}_{1,a} \square \{y_1\}$. Let $G = \hat{K}_{1,a} \square \hat{K}_{1,b}$, set $z_0 = (x_0, y_0) = \{x_0\} \square \{y_0\}$, $z_1 = (x_1, y_1) = \{x_1\} \square \{y_1\}$ and let $Q$ be the square induced by $z_1, z_0$ and their two common neighbors $z_2, z_3$. For a vertex $v \in (A(x_0) \cup B(y_0)) \setminus \{z_0\}$, let $v'$ be the unique neighbor of $v$ in $G - (A(x_0) \cup B(y_0))$ (see Fig 3).
Given a pairing of \( \min\{a, b\} \) (or \( \min\{a, b\} - 1 \)) terminals, our goal is to mate all terminals in \( A(x_0) \cup B(y_0) \) into the subgraph \( G' = G - (A(x_0) \cup B(y_0)) \cong K_{1,a} \sqcup K_{1,b} \) using pairwise edge disjoint mating paths and not using edges of \( G' \).

If \( z_0 \) is a terminal, then mate it into a free (non-terminal) vertex of \( Q \) by using a mating path of length 1 or 2. If every vertex of \( Q \) is a terminal, then mate \( z_0 \) with \( z_1 \) along the mating path \((z_0, z_2, z_1)\).

For \( s_i \in (A(x_0) \cup B(y_0)) \setminus \{z_0\} \), if \( s'_i \) is free (it is not a terminal or a mate), then let \( s'_i \) be the mate of \( s_i \). Let \( \alpha \) be the number of pairs \( w, w' \) such that \( w \in (A(x_0) \cup B(y_0)) \setminus \{z_0\} \) and precisely one of \( w \) and \( w' \) is a terminal (or mate).

Let \( s_\ell \in B(y_0) \setminus \{z_0\} \) and assume that \( s'_\ell \) is a terminal. If there exist an auxiliary vertex \( v \in B(y_0) \setminus \{z_0, z_3\} \) such that \( v \) and \( v' \) are both free, then we use the mating path of length 3 from \( s_\ell \) to \( v' \) through \( v \) as indicated in Fig.3. One finds a mating path of length 3 in the same way for \( s_\ell \in A(x_0) \setminus \{z_0\} \) using a free pair of auxiliary vertices \( v, v' \), where \( v \in A(x_0) \setminus \{z_0, z_2\} \). Assume that there are \( \beta \) pairs \( w, w' \) such that \( w \in (A(x_0) \cup B(y_0)) \setminus \{z_0\} \) and both \( w \) and \( w' \) are terminals (or mates).

For \( a \leq b \), there are at most \( a \) terminals in \( G \), hence \( \alpha + 2\beta \leq a \). This implies that

\[
\beta \leq a - (\alpha + \beta) \leq |B(y_0) \setminus \{z_0, z_3\}| - (\alpha + \beta) \leq |A(x_0) \setminus \{z_0, z_2\}| - (\alpha + \beta).
\]

Therefore, the number of the free pairs of auxiliary vertices in \( B(y_0) \setminus \{z_0, z_3\} \)

![Figure 3: Mates from \( T(3; 3) \sqcup T(4; 3) \) into \( K_{1,3} \sqcup K_{1,4} \)]
and also in $A(x_0) \setminus \{z_0, z_2\}$ is not smaller than $\beta$, hence the mating procedure succeeds.

Then we are done by induction, unless every vertex of $Q$ is a terminal. If this exceptional case happens, then all the $\min\{a, b\}$ mates and terminals are in $G' \cong K_{1,a} \Box K_{1,b}$ in such a way that the mate of $z_0$ and terminal $z_1$ coincide at a vertex of degree two of $G'$. We need a variation of a Lemma 3. Recall that $\Gamma$ is the set of vertices of the $(a + b)$-star subgraph of $K_{1,a} \Box K_{1,b}$.

**Lemma 9.** For $n \leq \min\{a, b\}$, let $X$ be a set of $n - 1$ distinct terminals in $K_{1,a} \Box K_{1,b}$ plus one additional terminal located at a vertex of degree two occupied by another terminal. Then there is a mapping $\phi : X \rightarrow \Gamma$ such that $|\phi(X)| = n$, $\phi(x) = x$, for $x \in X \cap \Gamma$, $\phi(x)$ is a $1$- or $3$-step boundary mate of $x$, for every $x \in X \setminus \Gamma$, and the mating paths determined by $\phi$ are pairwise edge-disjoint.

**Proof.** Let $\Gamma = \Gamma_a \cup \Gamma_b$, where $\Gamma_a$ induces a $K_{1,a}$ and $\Gamma_b$ induces a $K_{1,b}$. Let $s_i, s_j$ be the terminals located at the same vertex. We follow the proof of the original lemma. Observe that during the procedure when $s_i, s_j$ are to be mated into $\Gamma$, there is a free vertex in both $\Gamma_a$ and $\Gamma_b$, since $n - 2 < \min\{a, b\}$. Then we mate $s_i$ into $\Gamma_a$, and for $s_j$, we use a free vertex in $\Gamma_b$. □

After applying Lemma 9 the proof of the proposition concludes with the solution of the pairing in the star induced by $\Gamma$. □

It might be interesting to note that if one of the graphs in Proposition 8 is obtained from a star by subdividing one of its edges twice, then the path-pairability of their Cartesian products drops below 6, due to the cut condition.

**References**


3 THE PRODUCT OF TWO TREES DIFFERENT FROM A STAR


Appendix A.

In the tables below the vertices of $G(a, b) = K_{1,1} \Box K_{1,b}$ are arranged in a matrix such that each row $A(i) = \{(i,j) \mid 0 \leq j \leq b\}$ induces a copy of $K_{1,b}$ with center $(i,0)$, and each column $B(j) = \{(i,j) \mid 0 \leq i \leq a\}$ induces a copy of $K_{1,a}$ with center $(0,j)$. Let $\Gamma$ be the star induced by $A(0) \cup B(0)$.

**Proposition 10.** For $b \geq 2$, $pp(K_{1,1} \Box K_{1,b}) \leq \lceil (1 + b)/2 \rceil$.

**Proof.** For $b$ even, we define the terminal pairs $s_\ell, t_\ell$, $1 \leq \ell \leq b/2 + 1$, as follows:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$\cdots$</th>
<th>$s_{b/2-1}$</th>
<th>$s_{b/2}$</th>
<th>$s_{b/2+1}$</th>
<th>$t_{b/2+1}$</th>
<th>$t_{b/2}$</th>
<th>$t_{b/2-1}$</th>
<th>$\cdots$</th>
<th>$t_3$</th>
<th>$t_2$</th>
<th>$t_1$</th>
</tr>
</thead>
</table>

There is no solution, since each of the paths joining the terminals $s_{b/2}, t_{b/2}$ and joining the terminals $s_{b/2+1}, t_{b/2+1}$ cannot use a 'vertical' edge different from the edge $(0,0) - (1,0)$. For $b$ odd, take the pairs above for $b - 1$, then add a pair $s_{(b+1)/2}, t_{(b+1)/2}$ in the last column $B(b)$. $\square$

**Proposition 11.** $pp(K_{1,2} \Box K_{1,3}) \leq \lceil (2 + 3)/2 \rceil$.

**Proof.** The terminals of each pair $s_\ell, t_\ell$, $1 \leq \ell \leq 3$, below are at distance 4:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$t_3$</th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
</table>

Any path between the paired elements must use at least two edges of the boundary star $\Gamma$ which contains only 5 edges, hence there is no solution. $\square$

**Proposition 12.** For $b \geq 2$, $pp(K_{1,1} \Box K_{1,b}) = \lceil (b + 1)/2 \rceil$, furthermore, $pp(K_{1,2} \Box K_{1,3}) = 2$.

**Proof.** By Lemma 3, any set of at most $b + 1$ terminals of $G(1,b)$ has an injection into the path-pairable boundary star $\Gamma$. Then the first equality follows by Proposition 10.

By Lemma 3, any set of at most 4 terminals of $G(2,3)$ has an injection into the path-pairable star $\Gamma$. The second equality follows by Proposition 11. $\square$