On the Alspach conjecture

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Abstract

It has been conjectured by Alspach [2] that given integers \(n\) and \(m_1, \ldots, m_t\) with \(3 \leq m_i \leq n\) and \(\sum_{i=1}^t m_i = \binom{n}{2} (n \text{ odd})\) or \(\sum_{i=1}^t m_i = \binom{n}{2} - \frac{n}{2} (n \text{ even})\) then one can pack \(K_n\) (\(n\) odd) or \(K_n\) minus a 1-factor (\(n\) even) with cycles of lengths \(m_1, \ldots, m_t\). In this paper we show that if the cycle lengths \(m_i\) are bounded by some linear function of \(n\) and \(n\) is sufficiently large then this conjecture is true.

1 Introduction

The following is a conjecture of Brian Alspach [2].

**Conjecture 1** If \(m_1, \ldots, m_t\) are integers with \(3 \leq m_i \leq n\) and \(\sum_{i=1}^t m_i = \binom{n}{2} (n \text{ odd})\) or \(\sum_{i=1}^t m_i = \binom{n}{2} - \frac{n}{2} (n \text{ even})\) then one can pack \(K_n\) (\(n\) odd) or \(K_n\) minus a 1-factor (\(n\) even) with cycles of lengths \(m_1, \ldots, m_t\).

If we just require some circuits (Eulerian subgraphs) of sizes \(m_1, \ldots, m_t\) then the result is much easier and has been proved [5], however Conjecture 1 is far from being proved in general.

A number of special cases of this conjecture have been studied. The case when all the cycle lengths are the same has been investigated by Alspach, Gavlas and Marshall [3, 4]. Various cases when there are combinations of two or three distinct special cycle lengths have also be studied by Adams, Bryant, Khodkar and Fu [1, 6] and by Heinrich, Horak and Rosa [10]. Häggkvist has dealt with the case when all cycles are of even length and there are an even number of cycles of each length [9]. The conjecture has also been verified for \(n \leq 10\) by Rosa [12]. More recently, the author has verified the conjecture by computer for all \(n \leq 14\). In all these cases there are very strong assumptions made about the cycle lengths that can occur or the value of \(n\).

More generally, Caro and Yuster [7] have shown Conjecture 1 is true when \(n \geq N(L)\), where \(L\) is the maximum length of the cycles and \(N(L)\) is a function of \(L\) (see [7] and Theorem 19 below). Unfortunately the function \(N(L)\) is a very large and extremely rapidly increasing function of \(L\), in particular it grows faster than exponentially in \(L\). Our aim in this paper is to prove the conjecture when \(n\) is greater than some linear function of \(L\). For this we shall prove the following three theorems.
Theorem 1 Assume $n \equiv 2 \pmod{144}$. If $m_1, \ldots, m_t$ are integers with $72 \leq m_i \leq \left\lfloor \frac{n+37}{20} \right\rfloor$ and $\sum_{i=1}^t m_i = \left(\frac{n}{2} \right) - \frac{t}{2}$ then $K_n - I$ can be packed with cycles of lengths $m_1, \ldots, m_t$.

Theorem 2 Assume $n \geq N_1$ and $n \equiv 2 \pmod{144}$. If $m_1, \ldots, m_t$ are integers with $3 \leq m_i \leq \left\lfloor \frac{n+37}{20} \right\rfloor$ and $\sum_{i=1}^t m_i = \left(\frac{n}{2} \right) - \frac{t}{2}$ then $K_n - I$ can be packed with cycles of lengths $m_1, \ldots, m_t$.

Theorem 3 Assume $n \geq N_2$. If $m_1, \ldots, m_t$ are integers with $3 \leq m_i \leq \left\lfloor \frac{n-112}{40} \right\rfloor$ and $\sum_{i=1}^t m_i = \left(\frac{n}{2} \right) (n \text{ odd})$ or $\left(\frac{n}{2} \right) - \frac{t}{2} (n \text{ even})$, then one can pack $K_n$ (n odd) or $K_n - I$ (n even) with cycles of lengths $m_1, \ldots, m_t$.

The easiest result to prove is Theorem 1 and this forms the basis for the other two results. In Theorem 2 we have removed the lower bound on the lengths of the cycles and in Theorem 3 we have removed the congruence condition on $n$, however in both cases $n$ must be larger than $N_1$ or $N_2$ which are (very large) absolute constants. These theorems use the result in [7] mentioned above, but only for cycle lengths less than 72. All cycles of length at least 72 are handled separately, so the lower bound on $n$ is now linear in the size of the longest cycle. It is worth mentioning that it should be possible to improve the linear bound on the cycle lengths, perhaps to as much as about $n/2$. However, new ideas will be needed to pack cycles of lengths much closer to $n$. All three theorems require extensive computer verifications, which were performed using Visual Basic on a 150 MHz Pentium based PC.

We shall give a proof of Theorem 1 first. The ideas used are similar to those used to pack circuits in [5]. Our strategy is to pack cycles into sequences of linked octahedra. This is done in Section 2. The octahedra are then packed into $K_n - I$ in such a way that any $\left\lfloor \frac{n+37}{20} \right\rfloor$ consecutive octahedra are packed so that non-adjacent octahedra are vertex-disjoint. This last result is proved in Section 3 by finding “self-avoiding” trails of triangles in a Steiner Triple System in $K_{n/2}$ and doubling each vertex. Theorem 1 is then proved at the end of Section 3.

Theorem 1 is not the best possible with our methods. In particular the conditions $72 \leq m_i$ and $n \equiv 2 \pmod{144}$ can be weakened considerably without including any “sufficiently large” condition on $n$. However, improving this result introduces more technicalities than we wish to include here. Instead, we shall proceed with the proofs of Theorem 2 and Theorem 3 where we make the extra assumption that $n$ is very large. By including more packings and using the result of Caro and Yuster [7] we prove Theorem 2. This is done in Section 4. In Section 5 we remove the congruence condition on $n$ and prove Theorem 3. Finally in Section 6 we conclude by discussing possible improvements to these results.

2 Packing trails of Octahedra

As usual, write $K_n$ for a complete graph and $C_n$ for a cycle on $n$ vertices. Define $K_n'$ to be $K_n$ if $n$ is odd and $K_n$ minus a one-factor $I$ when $n$ is even. Note that every vertex of $K_n'$ has even degree and that $K_n'$ is the graph with the largest number of edges on $n$ vertices for which this is true. Write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of natural numbers including zero.

We shall define for some graphs initial and final links. These will be disjoint (ordered) pairs of vertices. If we have two such graphs $G_1$ and $G_2$, write $G_1.G_2$ for the edge-disjoint union of $G_1$ and
Theorem 4  If $\sum_{i=1}^{t} m_i = 12N$, $12N \geq 40L$ and $72 \leq m_i \leq L$ for $i = 1, \ldots, t$, then we can pack $O^N$ with cycles of lengths $m_1, \ldots, m_t$.

To prove this we shall first pack a single octahedron $O$ with various paths. Define packings $\{s\} = \{s_1, s_2, s_3, s_4\}$ with $\sum_{i=1}^{4} s_i = |E(O)| = 12$ as eight edge disjoint paths in $O$ as follows. For each $i = 1, \ldots, 4$ there are two vertex-disjoint paths each connecting an initial vertex of $O$ to a final vertex of $O$ with the sum of the lengths of these two paths equal to $s_i$. Since the two paths are vertex-disjoint, each of the four link vertices of $O$ will be an endpoint of one of the two paths.

We also define packings where some or all of the $s_i$ are replaced by sums $p_i + q_i$. In these cases there is one path of length $p_i$ joining the two initial link vertices and one path of length $q_i$ joining the two final link vertices (and as a special case, 0 denotes no path). The two paths do not need to be vertex-disjoint in this case. For simplicity we shall not use all possible packings at this stage. The lower bound of 72 in Theorem 1 and Theorem 4 can be reduced by using other packings as well (see Sections 4 and 6).

Lemma 5  The following packings of an octahedron exist:

\[
\begin{align*}
{2,2,4,4}, \quad & {2,3,3,4}, \quad & {3,3,3,3}, \quad & {2,3,3,2+2}, \\
{0+2,0+2,0+4,0+4}, \quad & {0+2,0+3,0+3,0+4}, \quad & {0+3,0+3,0+3,0+3}.
\end{align*}
\]
Proof. The result follows by inspection of the following diagrams:

Clearly any permutation (such as [4, 2, 2, 4]) or reversal (such as [2+0, 2+0, 4+0, 4+0]) also exists by symmetry. Note that there is no packing of the form [2, 2, 4, 2+2].

Define $S$ to be the set of 4-tuples $s = (s_1, s_2, s_3, s_4)$ for which $\sum_{i=1}^{4} s_i = 12$ and $2 \leq s_i \leq 4$ for all $i$. These are just the permutations of either (2, 2, 4, 4), (2, 3, 3, 4) or (3, 3, 3, 3). Also, write $0 = (0, 0, 0, 0)$. Lemma 5 implies that packings of the form $[s]$ and $[0+s] = [0+s_1, 0+s_2, 0+s_3, 0+s_4]$ exist for all $s \in S$.

By packing the octahedra using Lemma 5 and joining up the paths we shall get cycles of various lengths. This is best described by an example. Suppose we packed four linked octahedra as

$$O^4 = [0+3, 0+3, 0+3, 0+3]. [3, 3, 2, 2+2]. [2+2, 3, 3, 2]. [4+0, 3+0, 3+0, 2+0]$$

For each $i = 1, \ldots, 4$ join up the paths corresponding to the components $s_i$ or $p_i + q_i$ of each packing. By linking the $i = 1$ paths (0+3, 3+2, 4+0) we get two cycles of lengths $3 + 3 + 2 = 8$ and $2 + 4 = 6$ respectively. For $i = 2$ (0+3, 3, 3+0) we get one cycle of length 12, for $i = 3$ (0+3, 2, 3+0) we get one cycle of length 11 and for $i = 4$ (0+3, 2+2, 2+2) we get two cycles of lengths 5 and 6 respectively. The result is a packing of $C_5 + 2C_6 + C_8 + C_{11} + C_{12}$ into $O^4$.

Example of cycles packing $O^4$
We now need to describe an algorithm for choosing the packings for each octahedron in \( O^N \) so that we get the cycle lengths \( m_1, \ldots, m_t \) specified in the statement of the theorem. We shall construct the packing one octahedron at a time, so that at each step we may have some incompletely packed cycles. The unpacked parts of these cycles will form a graph \( \{a\} \) linked to the final link of the last octahedron packed so far. For example, after the first step in the example above we shall have a packing of \( C_8 + C_{12} + C_{11} + C_5 \) into \( O.\{5,9,8,2\} \). The next step packs this graph and one \( C_6 \) into \( O^2.\{2,6,6,4\} \). The next step packs this graph and the second \( C_6 \) into \( O^3.\{4,3,3,2\} \). The last step then packs this graph into \( O^4 \).

For \( a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \) define the following functions:
\[
\min a = \min_i a_i, \quad \max a = \max_i a_i, \quad \sum a = \sum_i a_i \quad \text{and} \quad k(a) = 2 \max a - 3 \min a.
\]

The function \( k(a) \) estimates the unevenness of the lengths and will be used later.

**Lemma 6** There exists a non-empty set \( A \subseteq \mathbb{N}^4 \) such that for any \( a \in A \) and any \( m \geq 27 \) we can pack \( O^m.\{a\} \) together possibly with the cycle \( C_m \) into some \( O^{m+1}.\{b\} \) with \( b \in A \). Moreover, \( a \in A \) for all \( a \) with \( \min a > 12 \) and \( \max a > 25 \).

**Proof.** Without loss of generality assume \( a_1 \leq a_2 \leq a_3 \leq a_4 \). We shall either pack the graph as \( O^m.\{s\}.\{a\} \) with \( b_i = a_i - s_i \) or as \( O^m.[2+2,s_2,s_3,s_4].\{b\} \) with \( b_i = a_i - s_i \) for \( i \geq 2 \), \( a_1 = 2 \) and \( b_1 = m - 2 \). In each case we use one of the packings of Lemma 5. The proof is by a computer verification. For this consider the \( a_i \) as elements of \( F = \{2, \ldots, n - 1, n^*\} \) where \( n^* \) is some fixed positive integer and \( n^* \) represents any number \( \geq n \). We construct sets \( A_j \) as a subset of \( F^4 \). Initially define \( A_0 \) as all \( a \in F^4 \) with \( a_1 \leq a_2 \leq a_3 \leq a_4 \) and inductively define \( A_{j+1} \) as all sequences \( a \in A_j \) that can be packed as above with \( b \in A_j \) for any \( m \geq 27 \). In other words, if \( a_1 > 2 \) then check if some permutation of \( (a_1 - s_1, a_2 - s_2, a_3 - s_3, a_4 - s_4) \) lies in \( A_j \) for some \( s \in S \).

If \( a_1 = 2 \) then for each \( m \geq 27 \) check if some permutation of \( (m - 2, a_2 - s_2, a_3 - s_3, a_4 - s_4) \) lies in \( A_j \) for some \( [2+2,s_2,s_3,s_4] \) packing of \( O \). If \( a_1 = n^* \) and \( b_i = a_i - s_i \) we need to check that \( b \in A_j \) for each choice of values \( b_i = n - s_i, n - s_i + 1, \ldots, n - 1, n^* \). If \( m - 2 \geq n \) then \( m - 2 \) is replaced by \( n^* \), hence only finitely many values of \( m \) need to be considered. The process terminates when \( A_{j+1} = A_j \). Since \( A_{j+1} \subseteq A_j \) and \( A_0 \) is finite, this will occur in finite time. If the final value of \( A_j \) is non-empty then we can take \( A \) as the set of \( a \in \mathbb{N}^4 \) which are permutations of 4-tuples \( a \in A_j \) where any value \( a_i \geq n \) is taken to be \( n^* \). For \( n = 26 \) this process does indeed terminate with a non-empty \( A_j \) and we are done. The largest value of \( a_1 \) with \( a \notin A_j \) and \( a_4 = 26^* \) is \( a_1 = 12 \) and occurs when \( a = (12, 12, 12, 26^*) \). Hence the final part also follows.  

Lemma 6 gives us the main inductive step, however we still need to solve two remaining problems—how to start and how to stop. Starting is easy as the following lemma shows.

**Lemma 7** If \( m_1, \ldots, m_4 \geq 26 \) then we can pack cycles \( C_{m_1}, \ldots, C_{m_4} \) into some graph of the form \( O.\{a\} \) with \( a \in A \). Here \( A \) is the set constructed in Lemma 6.

**Proof.** By Lemma 6, \( m = (m_1, m_2, m_3, m_4) \in A \), and by the proof of Lemma 6, \( m = a + s \) for some \( a \in A \) and \( s \in S \). Using Lemma 5 we can pack the cycles into \( [0+s_1, 0+s_2, 0+s_3, 0+s_4].\{a\} = O.\{a\} \).  

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Stopping however is much more difficult. We need to arrange the cycles and the packings so that the last four cycles finish simultaneously. For this we need the $a_i$ to be reasonably similar in length as the next two results show.

For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^4$ write $\mathbf{a} \rightarrow \mathbf{b}$ if there exists $\mathbf{s} \in \mathcal{S}$ such that

1. $\mathbf{a} = \mathbf{b} + \mathbf{s}$, i.e., $a_i = b_i + s_i$ for $i = 1, \ldots, 4$,
2. if $b_i < 0 \leq a_i$ then $b_i = -2$, $a_i = 2$ and $\mathbf{s}$ is a permutation of $(2, 3, 3, 4)$,
3. if $b_i = 0$ for some $i$ then $\mathbf{b} = (0, 0, 0, 0)$.

The use of negative numbers and conditions 2 and 3 above will become apparent in the proof of Lemma 12. Note that $b_i < 0 \leq a_i$ can hold for at most one value of $i$ since when this occurs $s_i = 4$ and $\mathbf{s}$ is a permutation of $(2, 3, 3, 4)$. In general we say $\mathbf{a} \Rightarrow \mathbf{b}$ if there is a sequence $\mathbf{a} = a_0, \ldots, a_s = \mathbf{b}$ for some $s \geq 0$ with $a_i \rightarrow a_{i+1}$ for all $0 \leq i < s$. Recall that $\mathbf{0} = (0, 0, 0, 0)$.

**Lemma 8** For all $\mathbf{a} \in \mathbb{Z}^4$, $\mathbf{a} \Rightarrow \mathbf{0}$ if and only if $\sum a = 12n$, $\min a \geq 2n$ and $\max a \leq 4n$ for some integer $n \geq 0$.

**Proof.** The “only if” is clear, since $\mathbf{a}$ must be the sum of $n$ terms in $\mathcal{S}$ and for each such term $\mathbf{s}$, $2 \leq s_i \leq 4$ for all $i$. It remains to prove the “if”.

We may assume $a_1 \leq a_2 \leq a_3 \leq a_4$. If $n = 0$ then $\mathbf{a} = \mathbf{0}$ and we are done. If $n = 1$ then $\mathbf{a} \in \mathcal{S}$ and we are done (taking $\mathbf{s} = \mathbf{a}$). Now assume $n \geq 2$. Let $\mathbf{s} = (2, 2, 4, 4)$ and consider $\mathbf{b} = \mathbf{a} - \mathbf{s}$. Clearly $\sum b = 12(n - 1)$. Also, $\min b \geq 2(n - 1)$ unless $a_3 \leq 2n + 1$ and $\max b \leq 4(n - 1)$ unless $a_2 \geq 4n - 1$. However, if $a_3 \leq 2n + 1$ then $12n = \sum a \leq 3(2n + 1) + (4n) = 10n + 3$, so $n < 2$, a contradiction. On the other hand, if $a_2 \geq 4n - 1$ then $12n = \sum a \geq (2n) + 3(4n - 1) = 14n - 3$, so again $n < 2$, a contradiction. Since $\min b > 0$ and $\mathbf{a} = \mathbf{b} + \mathbf{s}$ we have $\mathbf{a} \rightarrow \mathbf{b}$. Also $\mathbf{b} \Rightarrow \mathbf{0}$ by induction on $n$, and so $\mathbf{a} \Rightarrow \mathbf{0}$. \hfill \Box

**Corollary 9** If $\sum a = 12n > 0$ and $k(\mathbf{a}) \leq 0$ then $\mathbf{a} \Rightarrow \mathbf{0}$. In particular $O^r.\{\mathbf{a}\}$ can be packed into $O^{r+n}$.

**Proof.** Assume $a_1 \leq a_2 \leq a_3 \leq a_4$, so that $k(\mathbf{a}) = 2a_4 - 3a_1 \leq 0$. Hence $6a_1 \geq 4a_4 \geq \sum a = 12n$ and so $a_1 \geq 2n$. Also $3a_4 \geq a_4 + 3a_1 \geq \sum a = 12n$, so $a_4 \leq 4n$. The result follows from Lemma 8. The last part is clear since if $\mathbf{a} \Rightarrow \mathbf{0}$ then $\mathbf{a} = \mathbf{s}_1 + \ldots + \mathbf{s}_n$ and we can pack $O^r.\{\mathbf{a}\}$ as $O^r.[\mathbf{s}_1].[\mathbf{s}_2] \ldots [\mathbf{s}_n + \mathbf{0}] = O^{r+n}$. \hfill \Box

It remains to pack the cycles in a suitable order so that once we have used all the cycles we will have packed some $O^r.\{\mathbf{a}\}$ with $k(\mathbf{a}) \leq 0$. The next three lemmas show that if we pack four cycles of similar lengths then we can make the value of $k(\mathbf{a})$ decrease.

For $\mathbf{a} \in \mathbb{Z}^4$ define $\text{ord} \mathbf{a} \in \mathbb{Z}^4$ to be the non-decreasing rearrangement of $\mathbf{a}$. In other words, $\text{ord} \mathbf{a}$ is the 4-tuple obtained by putting the components of $\mathbf{a}$ in non-decreasing order, so for example $\text{ord}(5, 2, -7, 2) = (-7, 2, 2, 5)$.

**Lemma 10** If $\mathbf{a} \rightarrow \mathbf{b}$ then $\text{ord} \mathbf{a} \rightarrow \text{ord} \mathbf{b}$. 

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Proof. Since $a \rightarrow b$, $a = b + s$ for some $s \in S$. Assume $a_j < a_k$ and $b_j > b_k$, so that $b_k < b_j < a_j < a_k$. Then $s_k \geq s_j + 2$ so $s_k = 4$, $s_j = 2$ and $a_j = a_k - 1$, $b_j = b_k + 1$. If $b_k \leq 0 \leq a_k$ then either $b_k = 0$ or $b_k < 0 \leq a_k$. In the first case $b = 0$, contradicting the assumption $b_j > b_k$. In the second case $a_k = 2$, $b_k = -2$. But then $b_j = -1 < 0 < a_j = 1$ contradicting part 2 of the definition of $\rightarrow$. Hence $a_j, a_k, b_j$ and $b_k$ are all non-zero and have the same sign. Define $b'$ to be $b$ with $b_j$ and $b_k$ swapped. Then $a = b' + s'$ where $s'$ is $s$ with $s_j$ and $s_k$ replaced by 3. Clearly $s' \in S$ and so $a \rightarrow b'$. Repeating this process, we can assume the components of $b$ are in the same order as those in $a$. Hence there is a permutation $\pi$ of the numbers $1, \ldots, 4$ which simultaneously makes $(a_{\pi_1}, a_{\pi_2}, a_{\pi_3}, a_{\pi_4}) = \text{ord } a$ and $(b_{\pi_1}, b_{\pi_2}, b_{\pi_3}, b_{\pi_4}) = \text{ord } b$. By replacing $s$ with $(s_{\pi_1}, s_{\pi_2}, s_{\pi_3}, s_{\pi_4})$ we get $\text{ord } a \rightarrow \text{ord } b$. \hfill $\square$

Lemma 11 If $a \in \mathcal{A}$ then there exists some $b \in \mathbb{Z}^4$ with $a \Rightarrow b$, $\max b = -2$ and $k(b) \leq \max(38, 41 + k(a))$.

Proof. The proof is in two parts. The first is a computer verification of this lemma for all $a$ with $\max a \leq 43$. We may assume $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 43$. Construct an array $A$ of all $b$ with $-31 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 43$. (The values 31 and 43 are just the smallest values for which the proof works, larger values may of course be used.) Set $A(b) = k(b)$ for each element of this array with $b_4 = -2$. All other values are initially set to $+\infty$ (or some suitably large integer). Now, for each element of this array with $b_4 > -2$ in turn, set $A(b) = \min A(b')$ where the minimum is over all $b'$ with $a \rightarrow b'$ and $A(b')$ defined. If the elements of $A$ are processed in a suitable order (such as increasing lexicographical order), then the value of $\min A(b')$ will not be changed in any subsequent modification of $A$. Finally, the inequality $A(a) \leq \max(38, 41 + k(a))$ can be checked for each $a$ with $\min a \geq 2$, $\max a \leq 43$ and $a \in \mathcal{A}$, where the set $\mathcal{A}$ is constructed as in Lemma 6. Clearly $A(a)$ is an upper bound on the value of $k(b)$ for all $b$ with $a \Rightarrow b$ and $\max b = -2$.

The second part of the proof is an inductive proof to remove the upper bound of 43 on $\max a$. Assume therefore that $a_1 \leq a_2 \leq a_3 \leq a_4$ and $a_4 \geq 44$. We consider the case $a_1 \geq 21$ first.

If $a_2 \geq a_1 + 2$, then let $a' = a - (2, 4, 3, 3)$. Now $\min a' = a_1 - 2 > 12$, and $\max a' = a_4 - 3 > 25$, so $a \rightarrow a'$, $a' \in \mathcal{A}$ and $k(a') = 2(a_4 - 3) - 3(a_1 - 2) = k(a)$. By induction on $\max a$, $a \rightarrow a' \Rightarrow b$ with $\max b = -2$ and $k(b) \leq \max(38, 41 + k(a')) = \max(38, 41 + k(a))$ and we are done. If $a_3 \geq a_2 + 2$ then use the same argument with $a_1' = a - (2, 2, 4, 4)$. If $a_4 \geq a_3 + 4$ then use the same argument with $a_1' = a - (8, 8, 8, 12)$. Note that $(8, 8, 8, 12) = (3, 3, 2, 4) + (3, 3, 2, 4) + (2, 3, 3, 4)$ is a sum of elements of $S$ so $a \Rightarrow a'$. If none of these conditions hold then $a_4 \leq a_1 + 5$. In this case set $a' = a - (3, 3, 3, 3) \in \mathcal{A}$ and note that $k(a') = 2(a_4 - 3) - 3(a_1 - 3) \leq 13 - a_1 < -3$. Now $a \rightarrow a' \Rightarrow b$ with $k(b) \leq \max(38, 41 + k(a')) = 38 \leq \max(38, 41 + k(a))$.

Now assume $a_1 \leq 20$ and $a_4 \geq 44$. If $a_2 \geq 30$ then set $a' = a - (0, 4, 3, 3)$. Now $a'$ differs from $a$ only in terms that are at least 26. Hence by the proof of Lemma 6, $a' \in \mathcal{A}$. By induction $a' \Rightarrow b'$ with $\max b' = -2$ and $k(b') \leq \max(38, 41 + k(a'))$. Since the components of $a'$ are in non-decreasing order, we can replace each term $a_i'$ in $a'$ by $a_i \rightarrow a_i' \rightarrow \ldots \rightarrow a_i' = b'$ with ord $a_i'$ and $b'$ with ord $b'$. Clearly this new $b'$ still satisfies $b' = -2$ and $k(b') \leq \max(38, 41 + k(a'))$.

Since at each step the components are in non-decreasing order and since at most one component can jump across zero at each step, there must be some $a'' = a_i'$ with $a_i \Rightarrow a_i' \Rightarrow b'$ and $a'' < 0 < a''_2$. Hence $a \Rightarrow a'' \Rightarrow (0, 4, 3, 3) \Rightarrow a'' - (2, 0, 0, 0) \Rightarrow b = b' - (2, 0, 0, 0)$. However $k(a'') = k(a) - 6 \geq 2(44) - 3(20) - 6 > 0$ and $k(b) = k(b') + 6$, so $k(b) \leq (41 + k(a')) + 6 \leq \max(38, 41 + k(a))$ as required. A similar argument applies when $a_2 \leq 29$, $a_3 \geq 33$ with $a' = a - (0, 0, 4, 4)$ and
$b = b' - (2, 2, 0, 0)$ and when $a_3 \leq 32$, $a_4 \geq 44$ with $a' = a - (0, 0, 0, 12)$ and $b = b' - (8, 8, 8, 0)$. Since $a_4 \geq 44$ at least one of these conditions hold and we are done.

**Lemma 12** Assume $72 \leq m_2 \leq m_3 \leq m_4$, $\max a \leq m_4 - 2$, $a \in A$ and $3m_4 < 4m_1$. Then we can pack $O^r_{\bullet}(a)$ and cycles $C_{m_1}, \ldots, C_{m_4}$ into some $O^r_{\bullet}(a')$ with $a' \in A$ and $k(a') \leq \max(0, k(a) - m_1/3)$.

**Proof.** Use the previous lemma to find $b$ with $\max b = -2$, $a \Rightarrow b$ and $k(b) \leq \max(38, 41 + k(a))$. Now $k(a) \leq 2(m_1 - 2) - 3(2) = 2m_1 - 10$, so $k(b) \leq 2m_1 + 31$. Hence $\min b = -(k(b) + 4)/3 \geq (m_1 - 35)/3 - m_1 > 12 - m_1$. The method of packing is as follows. Let $a \rightarrow a_0 \rightarrow a_1 \rightarrow \ldots \rightarrow a_s = b$ and assume the components of each $a_i$ are in non-decreasing order. As long as the components are positive, we pack $O^r_{\bullet,i}(a_i)$ as $O^r_{\bullet,i}[s,i_{i+1}]$ as before where $s = a_i - a_{i+1}$. However, when a component in $a_i$ becomes negative in $a_{i+1}$ we start a new cycle $C_{m_j}$. We start $C_{m_4}$ when $(a_{i+1})_1$ becomes negative, $C_{m_3}$ when $(a_{i+1})_2$ becomes negative and so on. By the definition of $\Rightarrow$, only one of these occurs at a time and we can use (a permutation of) the $[2+2, 2, 3, 3]$ packing at the point when the $j$th component goes from $+2$ to $-2$ and we start the new cycle $C_{m_{5-j}}$. If we define $a'_j$ to be $a_i$ with $m_{5-j}$ added to each negative component $(a_i)_j$, then we have inductively defined packings into $O^r_{\bullet,i}(a'_i)$ for $0 \leq k \leq s$. When we finish we have a packing of $O^r_{\bullet}(a)$ and all four cycles into $O^r_{\bullet}(a')$ where $a' = a'_1 = (b_1 + m_1, b_2 + m_3, b_3 + m_2, b_4 + m_4)$ and $r = r' = r + s$. Since $\min b > 12 - m_1$, all the terms in $a'$ are positive (so we did not run out of edges in the cycles), and indeed, $\min a' > 12$, $\max a' \geq m_1 + b_4 = m_1 - 2 > 25$ so $a' \in A$. It remains to check the value of $k(a')$. Assume $\min a' = b_i + m_{5-i}$ and $\max a' = b_j + m_{5-j}$. If $i < j$ then

$$
\begin{align*}
k(a') &= 2b_j - 3b_i + 2m_{5-j} - 3m_{5-i} \\
&\leq k(b) - m_1 \\
&\leq \max(38 - m_1, k(a) + 41 - m_1) \\
&\leq \max(0, k(a) - m_1/3),
\end{align*}
$$

where in the last inequality we have used $m_1 \geq 72$. If $i > j$ then

$$
\begin{align*}
k(a') &= 2b_j - 3b_i + 2m_{5-j} - 3m_{5-i} \\
&\leq k(m) - b_1 \\
&\leq 2(4m_1/3) - 3m_1 + (k(b) + 4)/3 \\
&\leq (k(b) + 4 - m_1)/3 \\
&\leq \max(0, k(a) + 45 - m_1)/3).
\end{align*}
$$

If $k(a) \leq m_1/3$ then $k(a) + 45 - m_1 \leq 0$ and so $k(a') \leq 0$. If $k(a) \geq m_1/3$ then

$$
(k(a) + 45 - m_1)/3 \leq k(a) - 2m_1/9 + 15 - m_1/3 \leq k(a) - m_1/3.
$$

In all cases $k(a') \leq \max(0, k(a) - m_1/3)$. 

**Proof.** of Theorem 4

Order the cycle lengths in decreasing order. Look at the first (longest) four cycles. If the length of the fourth cycle is less than or equal to three quarters the length of the first, discard the first cycle. Otherwise remove the first four cycles and group them as $m_1$. Repeat this process until
there are fewer than four cycles remaining or until we have six groups $m_1, \ldots, m_6$. We now have $m_1, \ldots, m_k$ ($k \leq 6$) groups of cycles each consisting of four cycles of similar lengths, a set of discarded cycles $m_1, \ldots, m_s$ say, and the remaining cycles $m_{s+1}, \ldots, m_u$ say. First calculate the maximum length of discarded cycles. Since $m_{i+3} \leq 3m_i/4$ for $1 \leq i \leq s - 3$ and $m_1, m_2, m_3 \leq L$, it is easy to see that the total length of discarded cycles is bounded by an infinite geometric series with sum $3L/(1 - \frac{3}{4}) = 12L$. The total length of the grouped cycles and remaining cycles is at most $4kL$ and $(u - s)L$ respectively. If there are at most three remaining cycles then $12N < 12L + 4kL + 3L \leq 39L$ contradicting the assumption that $12N \geq 40L$. Hence $k = 6$ and $u \geq 4 + s \geq 4$. Pack all the cycles $m_1, \ldots, m_u$ into some $O^x, \{a\}$ with $a \in A$ using first Lemma 7 and then inductively using Lemma 6 until all the cycles have been used. These cycles should be packed in order of decreasing length. We shall now show that max $a \leq \min m_6 - 2$. If the path of length $a_i$ comes from packing one of the remaining cycles $C_{m_j}$, $(j > s)$, then $a_i \leq m_j - 2 \leq \min m_6 - 2$, so assume it comes from packing a discarded cycle $C_{m_j}$, $(j \leq s)$. By considering the geometric series above, the total of all the cycle lengths packed before $C_{m_j}$ is at most $12L - 12m_j$ if $j$ is 1 mod 3 and hence at most $12L - 10m_j$ in general. Therefore there are at most $L - 5m_j/6$ octahedra that have been fully packed before cycle $C_{m_j}$ is started. Also, $\sum a \leq 4L$ so $12r + 4L \geq |E(O^x, \{a\})| \geq 12N - 4kL \geq 16L$ and hence $r \geq L$. Since $a_i$ must reduce by at least two for each extra octahedron packed, $a_i \leq m_j - 2(r - L + 5m_j/6) \leq 2(L - r) - 2m_j/3 < 0$. This is clearly impossible, so no path $a_i$ is part of a discarded cycle.

Now $k(a) < 2 \max a < 2 \min m_6$. Pack the $m_i$ in reverse order ($m_6$ first) using Lemma 12. At each stage $\max a \leq \min m_i - 2$ and $k(a)$ is reduced by at least $\min m_i/3 \geq \min m_6/3$. Hence after six steps we have a packing of all the original cycles into some graph of the form $O^x, \{a\}$ with $a \in A$ and $k(a) \leq 0$. Since $\sum a$ is clearly divisible by 12, we can now pack this into $O^N$ by Corollary 9 as desired.

It is worth noting here that the proof of Theorem 4 can be strengthened to allow the packing of some $\{b\}$ with $b \in A$ at the start of the trail of octahedra:

**Corollary 13** Assume $b \in A$, $\max b \leq L$, $\sum b + \sum_{i=1}^t m_i = 12N$, $72 \leq m_i \leq L$ and $\sum_{i=1}^t m_i \geq 40L$. Then $\{b\}$ and the cycles $C_{m_1}, \ldots, C_{m_t}$ can be packed into $O^N$ with the initial link of $\{b\}$ packed into the initial link of $O^N$.

**Proof.** The proof is almost identical when we show that no discarded cycle $m_i$ remains partially packed when we start the $m_i$. For this let $L' = \sum b$. The total of all the cycle lengths packed before $C_{m_j}$ is now at most $L' + 12L - 10m_j$. Therefore there are at most $L'/12 + L - 5m_j/6$ octahedra that have been fully packed before cycle $m_j$ is started. As before $12r + 4L \geq 12N - 4kL$, but now $12N - 4kL \geq L' + 16L$ and hence $r \geq L + L'/12$. Since $a_i$ must reduce by at least two for each extra octahedron packed, $a_i \leq m_j - 2(r - L - L'/12 + 5m_j/6) \leq 2(L + L'/12 - r) - 2m_j/3 < 0$, which is a contradiction as before. Similarly, since $r \geq L$ and $\max b \leq L$, none of the original paths in $\{b\}$ will be left unpacked at this point.

**3 Self-Avoiding Trails of Triangles**

Define a $k$-self-avoiding trail of triangles in $K_n$ to be a sequence of triangles $T_1, \ldots, T_N$ in $K_n$ such that
1. the triangles \( T_1, \ldots, T_N \) are edge-disjoint,
2. for \( 1 \leq i < N \), \( V(T_i) \cap V(T_{i+1}) \neq \emptyset \),
3. for all \( i, j \), \( V(T_i) \cap V(T_j) \neq \emptyset \) implies either \( |i-j| \leq 1 \) or \( |i-j| \geq k \).

In other words, the triangles are linked together so that any subsequence containing at most \( k \) consecutive triangles forms a graph isomorphic to one of the form

A completely self-avoiding trail of triangles is a \( k \)-self-avoiding trail with \( k \) equal to the total number of triangles. We say that a \( k \)-self-avoiding trail packs \( K_n \) if the triangles also pack \( K_n \). In other words, the triangles form a Steiner Triple System for \( K_n \). The aim of this section is to prove the existence of self-avoiding trails with reasonably large \( k \) that pack \( K_n \), at least for some values of \( n \).

**Theorem 14** For all \( n \equiv 1 \mod 72 \) then there exists a \( \left\lfloor \frac{n+18}{20} \right\rfloor \)-self-avoiding trail of triangles which packs \( K_n \).

**Proof.** Write \( n = 24r + 1 \) and define the following set of triples of integers

\[
\mathcal{T} = \{ (2t, 10r - t, 10r + t) : t = 1 \ldots 2r \}
\]

\[
\cup \{ (1, 5r, 5r + 1) \}
\]

\[
\cup \{ (2t - 1, 6r - t + 1, 6r + t) : t = 2 \ldots r - 1 \}
\]

\[
\cup \{ (2r - 1, 6r, 8r - 1) \}
\]

\[
\cup \{ (2t - 1, 6r - t, 6r + t - 1) : t = r + 1 \ldots 2r - 1 \}
\]

\[
\cup \{ (4r - 1, 6r + 1, 10r) \}
\]

By inspection it is clear that for each triple \((a, b, c) \in \mathcal{T}\), \(a\), \(b\) and \(c\) are distinct, \(a + b = c\) and each integer from 1 to \(12r\) inclusive occurs in precisely one triple in \(\mathcal{T}\). As a result it is easy to see that

\[
\mathcal{S} = \{ (j, b+j, c+j) : j \in \mathbb{Z}/n\mathbb{Z}, (a, b, c) \in \mathcal{T} \}
\]

is a Steiner Triple System on the integers mod \(n = 24r + 1\). (This construction is based on a Skolem sequence.) See [14, 11, 13] for further details and other similar constructions.) Write the triangle \((j, b+j, c+j)\) as \(T_{a,j}\) where \((a, b, c) \in \mathcal{T}\). Such \(T_{a,j}\) exist for all \(a = 1, \ldots, 4r\) and \(j \in \mathbb{Z}/n\mathbb{Z}\).

Write \(r = 3m\) (so \(n = 72m + 1\)). We now construct a trail of these triangles (starting with \(T_2\)) as follows

\[
\begin{align*}
T_{6i+2} &= T_{6i+2, i} & (i, 10r - 2i - 1, 10r + 4i + 1) & i = 0, \ldots, 2m - 1 \\
T_{6i+3} &= T_{6i+3, 10r - 2i - 1} & (10r - 2i - 1, 16r - 5i - 2 - \delta, 16r + i + 1) & i = 0, \ldots, 2m - 1 \\
T_{6i+4} &= T_{6i+4, 16r + i + 1} & (16r + i + 1, 2r - 2i - 2, 2r + 4i + 2) & i = 0, \ldots, 2m - 1 \\
T_{6i+1} &= T_{6i+1, 16r - 2i} & (2r - 2i, 8r - 5i + 1 - \delta, 8r + i) & i = 1, \ldots, 2m - 1 \\
T_{6i} &= T_{6i, 8r + i} & (8r + i, 18r - 2i, 18r + 4i) & i = 1, \ldots, 2m - 1 \\
\end{align*}
\]
Here \( \delta = 0 \) if \( i < m \) and \( \delta = 2 \) if \( i \geq m \). The triangles include one triangle of the form \( T_{a,j} \) for each \( a \) other than \( a = 1, 2r - 1 \) and \( 4r - 1 \). Now add three more triangles corresponding to these values of \( a \).

\[
T_{12m-1} = T_{4r-1,4m} = (2m, 20m + 1, 32m) \quad \text{Note: } 2m = 2r - 2(2m - 1) - 2
\]

\[
T_{12m} = T_{2r-1,14m} = (14m, 32m, 38m - 1)
\]

\[
T_{12m+1} = T_{1,-m} = (-m, 14m, 14m + 1)
\]

In the last triangle \(-m\) is relatively prime to \( n = 72m + 1 \), so if we define \( T_{i+12m} \) to be \( T_i \) translated by \(-m\) (i.e., if \( T_i = T_{a,j} \) then \( T_{i+12m} = T_{a,j-m} \)), then the triangles \( T_2, \ldots, T_{12m+1} \) will be a permutation of the triangles \( T_{a,j} \) of the Steiner Triple System \( S \). It remains to show that it is \([\frac{2^{2/18}}{m}]\)-self avoiding. It is clearly enough by symmetry to show this for the subsequence \( T_2, \ldots, T_{24m-2} \). By inspection each \( T_i \) meets \( T_{i+1} \) at one vertex. The triangles \( T_2, \ldots, T_{12m-2} \) (and by symmetry \( T_{12m+1}, \ldots, T_{24m-2} \)) are completely self-avoiding since in the triangles above,

\[
-5i - 1 - \delta \in [-10m + 2, -6] \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
It remains to check the intersections of $T_{12m-1}$, $T_{12m}$ and $T_{12m+1}$ with the other triangles. The vertices of these three triangles which have not been dealt with already are $\{14m, 14m+1, 20m+1, 32m, 38m-1\}$. The possible intersections are listed below along with the smallest separations.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Closest Triangles</th>
<th>Minimum Separation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$14m = 8r - 5i' + 1 - \delta - m$</td>
<td>$i' = (9m - 1)/5$</td>
<td>$6(9m - 1)/5 - 2$</td>
</tr>
<tr>
<td>$14m + 1 = 8r - 5i' + 1 - \delta - m$</td>
<td>$i' = (9m - 2)/5$</td>
<td>$6(9m - 2)/5 - 2$</td>
</tr>
<tr>
<td>$20m + 1 = 8r - 5i + 1 - \delta$</td>
<td>$i = 4m/5$</td>
<td>$12m - 6(4m)/5$</td>
</tr>
<tr>
<td>$20m + 1 = 8r - 5i' + 1 - \delta - m$</td>
<td>$i' = 3m/5$</td>
<td>$6(3m)/5$</td>
</tr>
<tr>
<td>$32m = 10r + 4i+1$</td>
<td>No solutions</td>
<td></td>
</tr>
<tr>
<td>$32m = 10r + 4i' + 1 - m$</td>
<td>$i' = (3m - 1)/4$</td>
<td>$6(3m - 1)/4 + 2$</td>
</tr>
<tr>
<td>$38m - 1 = 16r - 5i' - 2 - \delta - m$</td>
<td>$i' = (9m - 3)/5$</td>
<td>$6(9m - 3)/5 + 3$</td>
</tr>
</tbody>
</table>

For $m \geq 1$, the smallest separation in any of the cases above is $6(3m)/5$. Hence the trail is $[6(3m)/5] = \lceil \frac{n+18}{20} \rceil$-self-avoiding.

Finally, we can now give the proof of Theorem 1.

**Proof.** of Theorem 1.

Let $n = 144m + 2$. Construct a $\lceil \frac{n/2+18}{20} \rceil$-self-avoiding trail of triangles that pack $K_{n/2}$ using Theorem 14. Now double up each vertex. Each edge is replaced by four edges, triangles are replaced by octahedra and $K_{n/2}$ becomes $K_n - I$. The trail of triangles become linked octahedra with the property that any $k$ consecutive octahedra with $k \leq \lceil \frac{n/2+18}{20} \rceil$ form a graph isomorphic to $O^k$. The total number of octahedra is $N = n(n-2)/24 = 6mn$. Since $72 \leq m_i \leq L = \lceil \frac{n+22}{20} \rceil < n$ and $12N \geq 72n > 40L$ we can pack this trail of octahedra with the cycles using Theorem 4. A cycle of length $m_i$ must be packed into at most $\lfloor \frac{L}{2} \rfloor$ consecutive octahedra since at least two edges of the cycle must occur in each of these octahedra. Since $\lfloor \frac{L}{2} \rfloor = \lceil \frac{n/2+18}{20} \rceil$ the vertices of the cycle will remain distinct when the octahedra are packed into $K_n - I$.

### 4 More Packing Results

We shall first generalise the notation used to indicate packings of the octahedron in Lemma 5. We shall extend the notation to cover other graphs where we have defined disjoint pairs of initial and final link vertices. Also, we shall no longer necessarily have exactly four terms of the form $s_i$ or $p_i + q_i$ and we shall include the possibility of whole cycles being packed in addition to the paths. So for example the packing $[2, 4] + 2C_3$ will denote a packing with two $C_3$’s and two pairs of paths from initial to final links (each pair being vertex disjoint, the first pair of total length 2 and the second of total length 4).

We now strengthen Lemma 5 to include more packings of the octahedron. In fact we shall include all useful packings.

**Lemma 15** The following packings of an octahedron exist. Indeed, in any packing of $O^N$ with
We shall denote these possibilities by cycles, each octahedra is packed in one of these forms (up to permutation and reversal).

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<tbody>
<tr>
<td>[2+2,0+2,0+4,0+4]</td>
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</tr>
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</table>

\[ \begin{array}{cccc}
1. & 2. & 3. & 4. \\
\| + 2C_6 & \| + C_6 + 2C_3 & \| + C_5 + C_4 + C_3 & \| + 3C_4 \\
\| + 4C_3 & [2, 2] + 2C_4 & [0+2, 0+2] + 2C_4 & [0+2, 0+2] + C_5 + C_3 \\
[3, 3] + C_6 & [3, 3] + 2C_3 & [0+2, 0+4] + C_6 & [0+2, 0+4] + 2C_3 \\
[0+3, 0+3] + C_6 & [0+3, 0+3] + 2C_3 & [2+2, 2] + C_6 & [2+2, 2] + 2C_3 \\
[3, 4] + C_5 & [0+3, 0+4] + C_5 & [0+2, 0+5] + C_5 & [0+2, 0+5] + 2C_3 \\
[0+2, 0+2, 0+2, 0+2] + C_4 & [2+2, 2+2] + C_4 & [3+3, 3] + C_3 & [3+4, 2] + C_3 \\
\end{array} \]

Proof. We make the following observation. If \( G \) is some graph with initial and final links defined as disjoint pairs of vertices and if \( G^n \) is packed with cycles, then the intersection \( H \) of a cycle with some component \( G \) must be one of the following:

1. A cycle in \( G \),
2. A path connecting the two initial link vertices of \( G \),
3. A path connecting the two final link vertices of \( G \),
4. Two vertex-disjoint paths both connecting an initial link vertex to a final link vertex.

We shall denote these possibilities by \( C_n \), \( n+0 \), \( 0+n \) and \( n \) respectively, where \( n \) is the total number of edges in \( H \). The set of all possible \( H \) can be constructed by computer as follows:

- List all subgraphs of \( G \) with maximum degree at most 2, even degree at all non-link vertices and the same parity of degree at the two initial link vertices and the same parity at the two final link vertices.
- Remove all \( H \) which strictly contain some other graph \( H' \) on this list and for which the set of vertices of degree one in \( H' \) is a subset of the vertices of degree 1 in \( H \). (This eliminates graphs that are unions of several possible \( H' \)’s.)
- Classify each \( H \) as “type \( C_n \)” if there are no degree 1 vertices, “type \( 0+n \)” if the final link vertices only have degree 1, “type \( n+0 \)” if the initial link vertices only have degree 1 and “type \( n \)” if all link vertices have degree 1. In each case \( n \) is the total number of edges in \( H \).
The set of all packings can then be constructed as the set of all partitions of the edges of $G$ into graphs on this list. We define the type of the packing as the collection of types of each component graph $H$. There are many possible partitions, but many are of the same type. For the $G = O$ we get just 109 types of decomposition. Of these, 27 are symmetric under reversal and the others form 41 pairs under reversal. This gives 68 types up to permutations and reversal, which we have listed above. For ease of notation, $n+0$ and $0+m$ components have been combined as $n+m$ in the table above.

The first 29 of the packings in Lemma 15 do not involve embedded cycles $C_n$ and these can be used to strengthen Lemma 6. First we generalize some more of our notation. In the notation \{a_1, a_2, a_3, a_4\} we shall include the possibility that some or all of the $a_i$ are zero. In this case the paths corresponding to $a_i$ do not exist. We also define variants of $\rightarrow$ and $\Rightarrow$ to correspond the the additional packings that we are using. We shall denote these modified versions as $\rightarrow^S$ and $\Rightarrow^S$ where $S$ denotes the set of packings used. The relation $\rightarrow^S$ will be defined as follows. If $a, b \in S$ then there must be some packing in $S$ that can be written as four terms $[t_1, t_2, t_3, t_4]$, each term $t_i$ being of the form $s$ or $p+q$ where $s > 0$ and $p, q \geq 0$. As before, if $p$ or $q$ is zero then they represent no path. Moreover,

1. If $t_i = s$ with $s > 0$ then $a_i = b_i + s$ and either $a_i, b_i > 0$ or $a_i, b_i < 0$.
2. If $t_i = p+q$ then $a_i = p$ and $b_i = -q$ (either or both $p$ and $q$ may be 0).

If $S$ contains the $[s], [s+0]$ and (all permutations of) the $[2+2, 2, 3, 3]$ packings of Lemma 5 then $\rightarrow^S$ and $\Rightarrow^S$ are the same as $\rightarrow$ and $\Rightarrow$ respectively. (Note that we must not include the $[0+s]$ packings since by definition $0 \rightarrow a$ is false for all $a$.) If $G$ is a graph for which we have defined disjoint initial and final links, then $\rightarrow^S$ and $\Rightarrow^S$ will denote $\rightarrow^S$ and $\Rightarrow^S$ with $S$ equal to the set of all packings of $G$ which do not include complete cycles. Note that $\rightarrow^S$ only differs from $\rightarrow$ when some of the components become or cross zero, however when this happens there are many more $\rightarrow^S$ relations than $\rightarrow$ relations. For example,

\[
\begin{align*}
(0, 0, 0, 0) & \rightarrow^S (-2, -2, -3, -5) \quad \text{Using } [0+2, 0+2, 0+3, 0+5] \\
(0, 0, 2, 3) & \rightarrow^S (-4, -3, 0, 0) \quad \text{Using } [0+4, 0+3, 2+0, 3+0] \quad \text{(a permutation of } [2+4, 3+3]) \\
(0, 0, 5, 9) & \rightarrow^S (-4, 0, 0, 6) \quad \text{Using } [0+4, 0+0, 5+0, 3] \quad \text{(a reversal of } [4+5, 3])
\end{align*}
\]

In order to simplify some of the proofs we shall define the following notation and algorithm. Let $A \subseteq \mathbb{N}^4$, $L \subseteq \mathbb{N}$ and let $S$ be a set of packings of a graph $G$ with disjoint initial and final links. Let $A(S, A, L)$ be the set of all elements $a \in \mathbb{N}^4$ such that for any choice of cycle lengths $m_1, \ldots, m_4 \in L$ the graph $O^r \{a\}$ can be packed together with cycles $m_1, \ldots, m_s$ for some $0 \leq s \leq 4$ into some graph of the form $O^r G \{b\}$ with $b \in A$. As in Lemma 6 we convert this into a finite problem suitable for computer calculation by identifying all numbers greater than or equal to some fixed number $n$. Let $u$ be the largest integer that occurs in a packing in $S$ as some $v+u$. For the octahedra this is at most 5. Let $F = \{-u, \ldots, n-1, n^*\}$ and let $\text{Fin}: \mathbb{N}^4 \rightarrow F^4$ be the function that sends components $a_i \geq n$ to $n^*$. With this we define the function $A_n(S, A, L)$ via the following algorithm.

- Let $A_n$ be the (finite) subset of $F^4$ consisting of all $a \in F^4$ such that $\text{Fin}^{-1}(a) \subseteq A$. 

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Lemma 16  There exists a non-empty set \( A' \subseteq \mathbb{N}^4 \) such that for any \( a \in A' \) and any \( m_1, \ldots, m_4 \geq 10 \) we can pack \( O^{r+1}.\{a\} \) together with cycles \( C_{m_1}, \ldots, C_{m_4} \) for some \( 0 \leq s \leq 4 \) into some \( O^{r+1}.\{b\} \) with \( b \in A' \). Moreover, \( 0 \in A' \) and \( a \in A' \) for all \( a \) with \( \min a > 4 \) and \( \max a > 12 \).

Proof.  The proof is similar to Lemma 6. The main difference is that some of the packings used start more than one new cycle at a time, so we may need up to four new cycles. Note in particular that we include the \([0+4]\) and \([s+0]\) of Lemma 5 or Lemma 15 which start or stop four cycles simultaneously. Define \( A'_0 = \mathbb{N}^4 \) and then inductively define \( A'_j = A'_n(O, A'_{j-1}, \geq 10) \). Eventually \( A'_j = A'_{j+1} \) and for this \( j \) we define \( A' = A'_j \). As a result, \( A' \subseteq A(O, A', \geq 10) \) and hence by definition we can pack \( O^r.\{a\} \) together with some given cycles of lengths at least 10 into some \( O^{r+1}.\{b\} \) with \( b \in A' \). A computer calculation shows that the set \( A' \) is non-empty when \( n = 13 \). The \( A' \) constructed by this method also satisfies the last conditions in the statement of the lemma so the result is proved.

Rather than prove a stronger analogue of Lemma 12, we shall show that with longer cycles we can go from graphs with \( a \in A' \) to graphs with \( a \in A \) where \( A' \) is the set constructed in Lemma 16 and \( A \) is the set constructed in Lemma 6.

Lemma 17  If \( a \in A' \) and \( m_1, \ldots, m_4 \geq 72 \) then we can pack \( O^r.\{a\} \) and cycles \( C_{m_1}, \ldots, C_{m_4} \) for some \( 0 \leq s \leq 4 \) into a graph of the form \( O^{r+7}.\{b\} \) with \( b \in A \).

Proof.  Using the set \( A \) constructed in Lemma 6 set \( A_0 = A \). Then for each \( j > 0 \) define \( A_j = A_n(O, A_{j-1}, \geq 72) \). Repeat this process until \( A' \subseteq A_j \). This indeed occurs when \( j = 7 \) and
Lemma 18  We have exact packings into some large of the two lists depending on whether \( s < r \). For the completed cycles packings of each packing of we use the packings of Lemma 15. For \( m \) constructed above for which the six \( C \) with \( b \), the form \( \sum b_i \). We put these packings into two lists, the first will contain the packings with \( b \neq 0 \) and the second will contain the packings of complete cycles only (\( b = 0 \)). For \( r = 1 \) we use the packings of Lemma 15. For \( r > 1 \) we take each such packing of \( O^{r-1} \) in turn and for each packing of \( O \) given by Lemma 15, we combine the two packings in every possible way to get packings of \( O^r \). Throw away any resulting packings that involve cycles or paths of lengths more than 9 and remove any repetitions of packings already achieved. We also discard the packing if the completed cycles \( \sum C_i \) contain a subset that can be packed completely into some \( O^s \) with \( 1 \leq s < r \) (and hence occur already in the second list). Any remaining packings are added to one of the two lists depending on whether \( b = 0 \) or not. This process eventually terminates since for large \( r \) all the packings contain subsets of cycles that can be packed exactly (this is not meant to be obvious, but running the computer program shows it to be true). It can now be checked that we have exact packings into some \( O^r \) of four \( C^3 \)'s, three \( C^4 \)'s, twelve \( C^5 \)'s, two \( C^6 \)'s, twelve \( C^7 \)'s six \( C^8 \)'s, or four \( C^9 \)'s respectively. Hence by packing any of these combinations we can assume we have at most eleven of each type of cycle. If our original collection of cycles contains a subset that can be packed exactly into some \( O^s \), then we pack these at the begining of the trail of octahedra. Hence by induction we can assume that no subset of cycles packs some \( O^s \) exactly. Now list each possible combination of cycles of length \( \leq 9 \) for which there is no subset that can be packed exactly into some \( O^s \), \( s \geq 1 \). This list is clearly finite. For each of these find some packing \( [0+b] + \sum C_i \) constructed above for which the \( C_i \)'s are a subset of the \( C_m \)'s. Check if the remaining \( C_m \)'s can packed by adding a graph \( \{ b' \} \) to \( O^r \), in other words if the lengths of the remaining cycles are the set of values \( b_i + b' \) for \( b' > 0 \). Now check that \( b + b' \) lies in the set \( A_{13,10}^\text{ext} \) constructed from \( \mathcal{A}' \) above. If it does, then we can add some of the cycles of length \( \geq 10 \) to get a packing into some \( O^r \). Running this algorithm shows that this succeeds except for the following combinations

\[ n = 26 \] and hence shows that for any \( a \in \mathcal{A}' \) we can pack \( O^r \{ a \} \) together with some cycles of length at least 72 into some graph of the form \( O^{r+7} \{ b \} \) with \( b \in \mathcal{A} \). Finally, it is clear that at most four cycles will be used since a cycle of length at least 72 cannot be packed completely into just seven linked octahedra. The result follows. 

We now need to deal with cycles of lengths less than 10. Define \( \tilde{O} \) to be the octahedron \( K_{2,2,2} \) but with different link vertices. The initial link will be the first class of this tripartite graph as before, but the final link will consist of two vertices one from each of the other two classes. As a result the final link will contain an edge.

![Diagram](https://via.placeholder.com/150)

**Lemma 18** If \( 3 \leq m_1, \ldots, m_s \leq 9 \) and \( m'_1, \ldots, m'_4 \geq 10 \) then we can pack cycles of lengths \( m_1, \ldots, m_s, m'_1, \ldots, m'_4 \) for some \( 0 \leq s' \leq 4 \) into some graph of the form \( O^r \{ a \} \) or \( O^{r-1} \{ a \} \).

**Proof.** Once again the result uses computer verification. We shall construct packings of \( O^r \) of the form \( [0+b] + \sum C_i \). We put these packings into two lists, the first will contain the packings with \( b \neq 0 \) and the second will contain the packings of complete cycles only (\( b = 0 \)). For \( r = 1 \) we use the packings of Lemma 15. For \( r > 1 \) we take each such packing of \( O^{r-1} \) in turn and for each packing of \( O \) given by Lemma 15, we combine the two packings in every possible way to get packings of \( O^r \). Throw away any resulting packings that involve cycles or paths of lengths more than 9 and remove any repetitions of packings already achieved. We also discard the packing if the completed cycles \( \sum C_i \) contain a subset that can be packed completely into some \( O^s \) with \( 1 \leq s < r \) (and hence occur already in the second list). Any remaining packings are added to one of the two lists depending on whether \( b = 0 \) or not. This process eventually terminates since for large \( r \) all the packings contain subsets of cycles that can be packed exactly (this is not meant to be obvious, but running the computer program shows it to be true). It can now be checked that we have exact packings into some \( O^r \) of four \( C^3 \)'s, three \( C^4 \)'s, twelve \( C^5 \)'s, two \( C^6 \)'s, twelve \( C^7 \)'s six \( C^8 \)'s, or four \( C^9 \)'s respectively. Hence by packing any of these combinations we can assume we have at most eleven of each type of cycle. If our original collection of cycles contains a subset that can be packed exactly into some \( O^s \), then we pack these at the beginning of the trail of octahedra. Hence by induction we can assume that no subset of cycles packs some \( O^s \) exactly. Now list each possible combination of cycles of length \( \leq 9 \) for which there is no subset that can be packed exactly into some \( O^s \), \( s \geq 1 \). This list is clearly finite. For each of these find some packing \( [0+b] + \sum C_i \) constructed above for which the \( C_i \)'s are a subset of the \( C_m \)'s. Check if the remaining \( C_m \)'s can packed by adding a graph \( \{ b' \} \) to \( O^r \), in other words if the lengths of the remaining cycles are the set of values \( b_i + b'_i \) for \( b'_i > 0 \). Now check that \( b + b' \) lies in the set \( A_{13,10}^\text{ext} \) constructed from \( \mathcal{A}' \) above. If it does, then we can add some of the cycles of length \( \geq 10 \) to get a packing into some \( O^r \{ a \} \). Running this algorithm shows that this succeeds except for the following combinations
of small cycles.

\[
(3,3,3,9) \quad (3,3,3) \quad (3,6,9) \quad (3,6) \quad (3,7,7,7) \quad (3,7,7,8) \\
(3,7,7) \quad (3,7,8) \quad (3,7,9,9,9) \quad (3,7,9,9) \quad (3,7,9) \quad (3,7) \\
(3,8,8,9,9) \quad (3,8,8,9) \quad (3,8,8) \quad (3,8,9,9,9) \quad (3,8,9,9) \quad (3,8,9) \\
(3,8) \quad (3,9,9,9) \quad (3,9,9) \quad (3,9) \quad (3)
\]

For these we modify the last octahedron to \( \tilde{O} \). Once this is done, these exceptional combinations can also be packed into \( \tilde{O}, \{\alpha\} \). Indeed, we only require the following two packings of \( \tilde{O} \) which are easily seen to exist.

\[
[0+1,0+2] + 3C_3, \quad [0+1,0+2,0+3,0+3] + C_3
\]

\[\square\]

We now need the following theorem, which is an immediate consequence of a result of Caro and Yuster (Theorem 4.1 of [7]). This result is proved using a theorem of Gustavsson [8] which in turn is a generalization of Wilson’s Decomposition theorem [16]. The upper bound on \( \epsilon(L) \) in Theorem 19 is just for convenience of use in the next corollary. In practice the \( \epsilon(L) \) given by [7] is extremely small. Write \( \delta(G) \) for the minimum degree of the vertices in \( G \).

**Theorem 19** There exist constants \( N(L) \) and \( \frac{1}{8} > \epsilon = \epsilon(L) > 0 \) depending only on \( L \) such that if \( G \) is a graph with \( |V(G)| \geq N(L) \), \( \delta(G) \geq (1-\epsilon)|V(G)| \) and every vertex is of even degree and if \( C_{m_1, \ldots, m_t} \) is a collection of cycles with \( 3 \leq m_i \leq L \) and \( \sum_{i=1}^{t} m_i = |E(G)| \) then there exists a packing of \( C_{m_1, \ldots, m_t} \) into \( G \).

**Corollary 20** If \( n \geq N(L) \), \( \sum_{i=1}^{t} m_i = |E(K'_n)| \), \( \sum_{m_i > L} m_i \leq \frac{n}{8}(\epsilon(L)n - 4) \) and \( 3 \leq m_i \leq \frac{n}{2} \) then we can pack cycles \( C_{m_1, \ldots, m_t} \) into \( K'_n \).

**Proof.** We start by packing the large cycles \( C_{m_i} \) with \( m_i > L \) into \( K'_n \) in such a way that no vertex is used too often. To be precise, the remaining degree at each vertex after removing these cycles will be at least \((1-\epsilon)n\) where \( \epsilon = \epsilon(L) \). We use a greedy algorithm. Assume we have packed some cycles already and we now need to pack \( C_{m_i} \). Let \( S \) be the set of vertices at which the degree is at least \((1-\epsilon)n + 2\) after removing the cycles already packed. Since each of the other vertices must meet more than \( \frac{1}{2}(en - 4) \) of the packed cycles, \( \frac{n}{8}(en - 4) \geq \sum_{m_i > L} m_i \geq \frac{1}{2}(n - |S|)(en - 4) \). Hence \(|S| \geq 3n/4\). Pick any vertex \( v_1 \) in \( S \) and inductively choose \( v_r \in S \) so that the vertex \( v_r \) and edge \( v_{r-1}v_r \) have not been used yet. At each stage there are at least \(|S| - (r - 1) - (en - 4) > \frac{3}{8} + 5 \) choices for \( v_r \). Finally for \( v_{m_i} \) we also need the edge \( v_{m_i}v_{m_i} \) to be unused. For this there are \(|S| - (m_i - 1) - 2(en - 4) > 9 \) choices. Add the cycle \( v_1, \ldots, v_{m_i} \) to our packing. Repeat this process until no more cycles are left. Now use Theorem 19 to pack the cycles of length \( \leq L \) into the the remaining edges of \( K'_n \).

**Proof.** of Theorem 2.

If \( n \geq 4 \) then \( m_i \leq \left\lfloor \frac{2n+37}{20} \right\rfloor \leq \frac{n}{2} \). By Corollary 20 with \( L = 71 \) we can assume there are more than \( \frac{n}{2}(\epsilon(71)n - 4) \) edges to be packed in cycles of length at least 72. As in Theorem 1 we pack a trail of octahedra. First we pack the cycles of length less than 10 using Lemma 18. We then use Lemma 16 to inductively pack those cycles of length between 10 and 71. By using at most four cycles of length \( \geq 72 \), we can ensure that all cycles of length less than 72 are used. (Since
Lemma 16 can use up to four cycles, we may need some longer cycles to guarantee the last few short cycles are packed, also, if there are no cycles of length between 10 and 71 we may need up to four longer cycles in Lemma 18.) Using four more cycles and Lemma 17 we get a packing of cycles into some \( O^r \{a\} \) with \( a \in A \) (or a similar graph in which one octahedron is replaced with \( O \)). Since we have used at most eight cycles of length \( \geq 72 \) and all cycles are of length at most \( \frac{n}{8} \), the remaining cycles will have total length of at least \( \frac{n}{8}(\epsilon(71)n - 4) - 8(\frac{n}{8}) \). Now use Corollary 13 to pack the remaining octahedra. This works provided \( \frac{n}{8}(\epsilon(71)n - 4) - 8(\frac{n}{8}) \geq 40(\frac{n}{2}) \) or equivalently \( n \geq 196/\epsilon(71) \). As in the proof of Theorem 1, packing the octahedra into \( K'_n \) gives the required packing of cycles. The only remaining complication is when one of the octahedra is replaced with \( \tilde{O} \) in Lemma 18. The only two edges that are in \( \tilde{O} \) but not \( O \) are edges joining doubled points, so are in the 1-factor \( I \) in \( K'_n = K_n - I \). Hence by slightly modifying this 1-factor we can pack one \( \tilde{O} \) in place of one of the \( O \)’s. The value of \( N_1 \) can be taken as \( \max(N(71), 196/\epsilon(71)) \).

\[ \square \]

5 Proof of Theorem 3

In this section we shall remove the congruence condition on \( n \) to obtain Theorem 3. To do this we will divide the vertices \( V \) of \( K'_n \) as \( V = V_0 \cup V_1 \) where \( |V_0| = 2m_2 \equiv 2 \mod 144 \) and \( |V_1| = n_1 \) is small. For technical reasons some small values of \( n_1 \) are not allowed and so we shall insist that \( 6 \leq n_1 \leq 149 \). The edges of \( K'_n \) now consist of the edges of \( K'_2m_0 \), the edges of \( K'_n \), the edges of a bipartite graph \( K_{2m_0,n_1} \) and (if \( n \) is odd) the missing 1-factor \( I \) of \( K'_2m_0 \).

Define the graphs \( O_{[a,b]} \) as \( O \) with two paths of lengths \( a \) and \( b \) joining the two non-link (middle) vertices of \( O \).

\[ O_{[1,2]} \quad O_{[2,2]} \quad O_{[2,3]} \]

The general strategy is as follows. To include the edges not in \( K'_{2m_0} \) we shall add “detours” to the octahedra packing \( K'_{2m_0} \). If \( n_1 \) is odd we shall replace some octahedra with \( O_{[1,2]} \). The paths of length 1 give us the missing 1-factor of \( K'_{2m_0} \) and the paths of length 2 join vertices in \( K'_{2m_0} \) to one of the vertices of \( K'_n \). Replacing some more octahedra with \( O_{[2,3]} \) we can use up the edges in \( K'_n \). Each path of length 3 will use one edge in \( K'_n \) and two edges in the bipartite graph. Finally, replacing yet more octahedra with \( O_{[2,2]} \) will use up the remaining edges of the bipartite graph. Hence, provided we choose the modified octahedra carefully, we shall be able to use up all the edges in \( K'_n \). Unfortunately, we now need to pack cycles into these modified octahedra, and for very short cycles we shall need to introduce several alternative modifications to the octahedra.

**Lemma 21** Assume \((a, b) = (1, 2), (2, 2) \) or \((2, 3) \). If \( a \in A \) and \( m_1, \ldots, m_{32} \geq 12 \) then we can pack \( O^r \{a\} \) and cycles \( C_m \) for some \( 0 \leq s \leq 32 \) into a graph of the form \( O^r + 7. O_{[a,b]} \{b\} \) with \( b \in A \).

**Proof.** First we construct the packings of \( O_{[a,b]} \) corresponding to the table in Lemma 15 above for \( O \). The algorithm used is identical to that of Lemma 15. We shall not list all the packings.
here as there are somewhat more of them than for $O$. Using the set $A'$ constructed in Lemma 16, construct $A'_0 = A_n(O_{[a,b]}, A', \geq 12)$. Then for $j = 1, \ldots, 7$ define $A''_j = A_n(O, A'_{1-j}, \geq 12)$. Finally for $n = 15$ and each choice of $(a, b)$ it can be checked by computer that $A' \subseteq A''_7$. As a result, we can pack $O^{+r}. \{a\}$ and cycles $C_m$, into $O^{+r+7}. O_{[a,b]} . \{b\}$ for some $b \in A'$. Clearly at most 32 cycles $C_m$, will be required. The result follows.

We now need to deal with cycles of length less than 12. Since the $O_{[2,3]}$’s are rare, we shall only need to pack $O_{[1,2]}$’s and $O_{[2,2]}$’s. Unfortunately we shall also need to consider some other graphs when the cycle lengths are very short. These other graphs are shown below. In each of these the initial link is the leftmost pair of vertices and the final link is the rightmost pair. The graph $W$ is obtained by adding three vertices $v_1, v_2, v_3$ to $O$ and joining them to the initial and final link vertices. Two additional edges are also added, one between the two initial link vertices and one between the two final link vertices. The graphs $W_{222}$ and $W_{33}$ are obtained by removing a $C_6$ from $W$. The graph $W_{222}$ has the missing $C_6$ meeting each of the vertices $v_1, v_2, v_3$ so that there are paths of length 2 missing from $v_1$ to $v_2$, from $v_2$ to $v_3$ and from $v_3$ to $v_1$. The graph $W_{33}$ has two paths of length 3 removed between $v_1$ and $v_3$. The graphs $W$ and $W_{33}$ are symmetric under reversal (interchange of initial and final links), however $W_{222}$ is not. When we write $W_{222}$ in the following lemma we mean than some choice of either $W_{222}$ or its reversal will make the result true.

Define initial and final links of $K'_8 = K_{2,2,2,2}$ as two distinct vertex classes. By symmetry it does not matter which two classes are used or the order of the two vertices in each link. Define initial and final links of $K_7$ as any two disjoint pairs of vertices. Once again, by symmetry it does not matter which pairs are chosen.

**Lemma 22** Let $3 \leq m \leq 11$. Assume $G$ is a graph of the form $G_1.G_2 \ldots G_8$ and one of the following conditions hold:

1. for all $i$, $G_i \in \{O, O_{[2,2]}, O_{[1,2]}\}$ and $m \in \{9, 10, 11\}$,
2. for all $i$, $G_i \in \{O, O_{[2,2]}, W_{222}, W_{33}\}$ and $m \in \{4, 6, 8\}$,
3. for all $i$, $G_i \in \{O, O_{[2,2]}, W\}$ and $m \in \{6, 7\}$,
4. for all $i$, $G_i \in \{O, K_7, K'_8\}$ and $m \in \{3, 5\}$.

Assume also that if $G_i \neq O$ then $i \geq 6$ and if $G_i, G_j \neq O$ then either $i = j$ or $|i - j| \geq 6$ (or $i \geq 12$ and $|i - j| \geq 12$ respectively when $m = 11$), so the non-$O$ graphs are well separated from each other and from the beginning of the sequence. If $|E(G)|$ is divisible by gcd$(m, 12)$, then we can pack some graph $G.O^{+r}$ with $0 \leq r \leq 10$ completely with cycles of length $m$. Also, if $m \in \{3, 4, 6\}$ we can take $r = 0$. 

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Proof. For each $m$ we define two finite sets $A_0$ and $A_+$ with $A_0 \subseteq A_+ \subseteq \mathbb{N}^4$ as follows:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$A_+$ (underlined elements lie in $A_0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>{(0,0,0,0), (8,8,8,8), (5,5,5,5), (2,2,2,2), (0,0,9,9), (7,7,7,7), (4,4,4,4), (0,0,2,2), (9,9,9,9), (6,6,6,6), (3,3,3,3)}</td>
</tr>
<tr>
<td>10</td>
<td>{(0,0,0,0), (6,6,8,8), (2,2,6,6), (4,4,8,8), (2,2,4,4), (0,0,2,3), (0,0,6,7), (2,3,8,8), (0,0,4,5), (2,3,6,6)}</td>
</tr>
<tr>
<td>9</td>
<td>{(0,0,0,0), (6,6,6,6), (3,3,3,3), (0,0,2,2), (7,7,7,7), (4,4,4,4), (0,0,4,4), (0,0,7,7), (3,3,7,7)}</td>
</tr>
<tr>
<td>8</td>
<td>{(0,0,0,0), (4,4,6,6), (0,0,4,4), (2,2,4,4)}</td>
</tr>
<tr>
<td>7</td>
<td>{(0,0,0,0), (4,4,4,4), (0,0,2,2), (5,5,5,5), (2,2,2,2), (0,0,5,5), (3,3,3,3)}</td>
</tr>
<tr>
<td>6</td>
<td>{(0,0,0,0), (0,0,2,2), (0,0,4,4)}</td>
</tr>
<tr>
<td>5</td>
<td>{(0,0,0,0), (2,2,2,2), (0,0,3,3), (0,0,2,2), (3,3,3,3)}</td>
</tr>
<tr>
<td>4</td>
<td>{(0,0,0,0)}</td>
</tr>
<tr>
<td>3</td>
<td>{(0,0,0,0)}</td>
</tr>
</tbody>
</table>

The set $A_0$ is defined as the subset of $A_+$ which occurs underlined in the above table. We first check that for each $m$ and any $a \in A_+$, the graph $O^t.a$ can be packed with possibly some $C_m$'s into some graph $O^{t+1}.b$ with $b \in A_+$. In terms of the notation above, it is enough that $A_+ \subseteq A_m(O,A_+,\{m\})$. This holds for $m \geq 7$, however, for $m \leq 6$ we also need to use some packings with embedded $C_m$'s in the octahedra. Define $S_{O,m}$ to be the set of packings of $O$ together with the set of packings of $O$ with one or more $C_m$'s removed. So, for example, since there is a $[2,2] + 2C_4$ packing of $O$, the packing $[2,2]$ will occur in $S_{O,4}$. It is now clearly sufficient that $A_+ \subseteq A_m(S_{O,m},A_+\{m\})$. This holds for all $m$ and $A_+$ in the table above. We also check that with enough additional $C_m$'s we can pack $O^t.a$ into some $O^{t+r}.b$ with $b \in A_0$ and $0 \leq r \leq 10$. Both these facts can be verified by hand easily from the above table and the packings of Lemma 15.

Now let $H$ be one of the other possible graphs listed above. Let $S_{H,m}$ be the set of packings of $H$, where we are allowed to remove $C_m$'s from $H$. For example, the $[0+4,0+4] + 2C_6$ packing of $W_{222}$ exists, so we can include $[0+4,0+4]$ in $S_{H,m}$ when $H = W_{222}$ and $m = 6$. For $H = W_{222}$ we include in $S_{H,m}$ packings of both $W_{222}$ and its reversal. Define $A'_j = A_m(S_{H,m},A_+,\{m\})$. Then define $A'_j = A_m(S_{O,m},A'_{j-1},\{m\})$ for $j \geq 2$. It can be checked (by computer) that $A_+ \subseteq A'_2$ always, and $A_+ \subseteq A'_r$ except in the case $m = 11$. Note that the cases when $H = W$ are trivial, since one can remove two $C_7$'s from $W$ to get $O$ or one $C_6$ from $W$ to get either $W_{222}$ or $W_{333}$.

As a consequence of these results, if $a \in A_+$, then we can pack $\{a\}$ and $C_m$'s into graphs of the form $O.\{b\}$ and $O^r.H.\{b\}$ for any $r \geq 5$ ($r \geq 11$ if $m = 11$), any $H$ and some $b \in A_+$ with the initial link of $\{a\}$ mapped to the initial link of these graphs. Hence by induction (and using $0 \in A_+$) we can pack $G.\{b\}$ completely with $m$-cycles for some $b \in A_+$. Finally, by adding more octahedra, we can pack $G.O^r.\{b\}$ with $m$-cycles for some $b \in A_0$. For $m \in \{3,4,6\}$, $A_0 = A_+$ so we can take $r = 0$. By assumption, $\gcd(m,12)$ divides $|E(G)|$. Since $\gcd(m,12)$ also divides $|E(O)|$ and $|E(C_m)|$, it must divide $|E(\{b\})| = \sum b$. By inspection of the table above, it is clear that in each case $b = 0$ and we are done.

For ease of checking, the following table lists the packings of $W$, $W_{222}$, $W_{333}$, $K_7$ and $K_8$ needed in this proof. They can all be generated by the algorithm of Lemma 15 or constructed by hand.
The required packings of $O_{[2,2]}$, $O_{[1,2]}$ are much more numerous, so are not listed here.

$$
W \quad O + 2C_7 \quad W_{222} + C_6 \text{ or } W_{33} + C_6
$$

$$
W_{222}, W_{33} \quad [\quad + 5C_4 \quad [0+4, 0+4] \quad + 2C_6 \quad [0+2, 2+2] \quad + 2C_6
\quad [0+4, 0+4, 2, 2] \quad + C_8 \quad [0+4, 0+4, 0+2, 0+2] \quad + C_8
$$

$$
K_7 \quad [\quad + 7C_3 \quad [1+1, 2+2] \quad + 3C_5
\quad [2+3, 2+3, 0+3, 0+3] \quad + C_5 \quad [0+3, 0+3] \quad + 3C_5
$$

$$
K'_8 \quad [\quad + 8C_3 \quad [0+2, 0+2] \quad + 4C_5
\quad [3+2, 3+2, 0+2, 0+2] \quad + 2C_5 \quad [3+3, 3+3, 3+3, 3+3]
$$

Note that in Lemma 22, no cycle $C_m$ will meet more than one $G_i$ with $G_i \neq O$ since these $G_i$ are separated by at least $m/2$ octahedra. It is also worth noting that for each $m$ with $3 \leq m \leq 11$, one can pack $12/gcd(m, 12)$ $m$-cycles into $O^{m/gcd(m,12)}$ except for the case $m = 8$ when we need 6 $C_8$’s to pack $O^4$.

We now need to modify the octahedra as described at the beginning of this section so that we pack the whole of $K'_n$. For this we need some extra properties of the self-avoiding trail of triangles constructed in Section 3. In analogy to the situation that occurs when we double vertices, the vertices of each triangle in Theorem 14 that meet the previous or subsequent triangle will be called link vertices and the other vertex will be called the non-link vertex or midvertex of the triangle.

**Lemma 23** There exists a constant $c_1$ such that for all $n \equiv 1 \mod 72$ and any vertex $v$ of $K_n$, any $c_1n$ consecutive triangles in the trail of triangles constructed in Theorem 14 contains a triangle with $v$ as its midvertex.

**Proof.** We shall just consider the midvertices of the triangles $T_{12mj+6i+2}$. These are of the form $10r + 4i + 1 - mj \mod n$ with $0 \leq i < 2m$. Taking any $12m$ consecutive triangles $T_k$ it is clear that for some $j$ we have triangles of the form $T_{12mj+6i+2}$ for at least $m$ consecutive values of $i$. Hence any $12m$ consecutive triangles contain every fourth vertex in a contiguous block of length $4m$ as a midvertex of one of these triangles. Since $T_{12m+k}$ is a translate of $T_k$ by $-m$ and $n = 72m + 1$, it is now clear that any consecutive set of $72(12m)$ triangles $T_k$ contains at least one out of every four consecutive vertices as a midvertex. But $T_{72(12m)+k}$ is a translate of $T_k$ by $-72m \equiv 1 \mod n$, so it is now clear that any consecutive sequence of $4(72)(12m) < 48n$ triangles contains every vertex as a midvertex. The result follows with $c_1 = 48$. $\Box$

**Lemma 24** Let $T_2, \ldots, T_{12mn+1}$ be the sequence of triangles constructed in Theorem 14 and assume $r \leq \frac{m-9}{6}$. Then there exists disjoint collections $S_1, \ldots, S_r$ of these triangles with the following properties:

1. all triangles in $\cup S_i$ lie in the first $4mn$ triangles of the sequence,
2. no two triangles in $\cup S_i$ are closer than 6 apart in the sequence,
3. for each $i$, the triangles of $S_i$ are vertex-disjoint,
4. there are at least $24m - 144r - 143$ triangles in each $S_i$. 

21.
Once again we consider just the translates of the triangles $T_{6i+2}$ as in Lemma 25, and consider the translates $T_{12mj+6i+2}$ of the triangle $T_{6i+2} = (i, 30m - 2i - 1, 30m + 4i + 1)$ of Theorem 14. All such triangles are at least 6 apart in the sequence, so condition 2 will hold. If we fix $i$ and if two translates $T_{12mj+6i+2}$ and $T_{12m(j+k)+6i+2}$ meet, then one of the following equations must be satisfied:

$$30m - 2i - 1 \equiv i \pm mk \mod n,$$

$$30m + 4i + 1 \equiv i \pm mk \mod n,$$

$$30m - 2i - 1 \equiv 30m + 4i + 1 \pm mk \mod n.$$  

Multiplying by 72 and using $72m \equiv -1 \mod n$ we get

$$\pm k \equiv 102 + 72(3i) \text{ or } 42 + 72(3i) \text{ or } n - 144 - 72(6i) \mod n$$

Now $24m - 144r \leq 72(3i) \leq 24m + 72r < n/2 - 102$, so

$$k \geq \min(24m - 144r + 102, 24m - 144r + 42, 24m - 144r - 143) = 24m - 144r - 143$$

Hence if we take $24m - 144r - 143$ consecutive triangles of this form they will be vertex-disjoint. Finally, any $4mn$ consecutive triangles of the trail contain at least $n/3 = 24m$ such triangles, so we can assume they all lie within the first $4mn$ triangles. Since there are $r$ distinct values of $i$ possible, we get $r$ disjoint collections of such triangles. \hfill \square

**Lemma 25** Let $T_2, \ldots, T_{12mn+1}$ be the sequence of triangles constructed in Theorem 14 and assume $m \geq 36$. Then there exists disjoint subsets $S_i$ of these triangles for $0 \leq i \leq \frac{m-10}{4}$ with the following properties:

1. all triangles in $\cup S_i$ lie in the first $6mn + 211n$ triangles of the sequence,
2. no two triangles in $\cup S_i$ are closer than 12 apart in the sequence,
3. no two triangles in any individual $S_i$ are closer than 5m apart in the sequence,
4. the mid-vertices of the triangles in each $S_i$, $i \geq 1$ are distinct and enumerate all the vertices of $K_n$,
5. all the link vertices of triangles in $S_0$ are distinct and enumerate all but at most 3827 of the vertices of $K_n$,
6. if $v$ occurs as a link vertex of $T \in S_0$ and as a mid-vertex of $T' \in S_1 \cup S_2$ then $T'$ occurs after $T$ in the sequence.

**Proof.** Once again we consider just the translates of the triangles $T_{6i+2} = (i, 30m - 2i - 1, 30m + 4i + 1)$. Let $i_0 = \lfloor \frac{9m - 28}{8} \rfloor$, $j_0 = 288i_0 - 4n$, $i_1 = \lfloor \frac{m-10}{2} \rfloor$. Define the sets $S_i$ as

$$S_i = \{ T_{12mj+6(2i-2)+2} : j = 0, 1, \ldots, j_0 - 1 \}$$

$$\cup \{ T_{12m(j-j_0)+6(i_0+2i-2)+2} : j = j_0, \ldots, n - 1 \} \quad (i \geq 1)$$

$$S_0 = \{ T_{12mj+6i_1+2} : j = 0, \ldots, 36m - 1914 \}$$
2. It is enough to show that the numbers $2i - 2, i_0 + 2i - 2$ and $i_1$ are all at least 2 apart and between 0 and $2m - 2$. Since $1 \leq i \leq r$ it is enough that

$$2r - 2 \leq i_1 - 2, \quad i_1 \leq i_0 - 2 \quad \text{and} \quad i_0 + 2r - 2 \leq 2m - 2.$$ 

Substituting the definitions of $i_0$ and $i_1$ gives the following sufficient conditions

$$r \leq \frac{m - 10}{4}, \quad 4m - 40 \leq 9m - 44 \quad \text{and} \quad r \leq \frac{7m}{16}.$$ 

All are satisfied when $1 \leq r \leq \frac{m-10}{4}$.

1. Note that $36m - 1264 \leq j_0 \leq 36m - 1012$, so if $T_k \in S_i$ and $i \geq 1$ then

$$k \leq 12m(\max(j_0 + 1, n - j_0) \leq 12m(36m + 1265) \leq 6mn + 211n.$$ 

Also, $j_0 > 0$ when $m \geq 36$. The result for $S_0$ is clear.

3. Is clear for $S_0$, and is true for $S_i, i > 0$ provided $5m \leq 6i_0 \leq 12m - 5m$. However, $m - 4 \leq i_0 \leq 9m/8$ and $5m \leq 6(m - 4), 6(9m)/8 \leq 12m - 5m$ when $m \geq 36$.

4. The midvertices are $30m + 8(i - 1) + 1 - mj$ for $j < j_0$ or $30m + 8(i - 1) + 4i_0 + 1 - m(j - j_0)$ for $j \geq j_0$. However $4i_0 + mj_0 \equiv 0 \mod n$ so both these expressions are $30 + 8(i - 1) + 1 - mj \mod n$. Since $m$ is relatively prime to $n$, this enumerates all the numbers mod $n$ as $j$ ranges from 0 to $n - 1$.

5. The link vertices are $i_1 - mj$ and $30m - 2i_1 - 1 - mj$. As $j$ runs through the numbers mod $n$, it is clear that all the vertices $i_1 - mj$ are distinct mod $n$. Similarly the vertices $30m - 2i_1 - 1 - mj$ are all distinct mod $n$. If $i_1 - mj \equiv 30m - 2i_1 - 1 - m(j \pm k)$ mod $n$ then $\pm k \equiv 72(3i_1 + 42 \cdot 3m/2 | m) - 466$ and $|k| \geq 36m - 574$. Hence all the link vertices are distinct. Since there are $36m - 1913$ triangles, the link vertices enumerate all but $n - 2(36m - 1913) = 3827$ of the vertices of $K_n$.

6. Consider a triangle in $S_0$. The link vertices are $i_1 - mj$ and $30m - 2i_1 - 1 - mj$ and these are the midvertices $30m + 8(i - 1) + 1 - mj'$ in $S_i$ for some $j'$ by part 4. If $i_1 - mj \equiv 30m + 8(i - 1) + 1 - mj'$ and $i = 1$ or 2 then it can be checked that $34 \leq j' - j - j_0 \leq 826$. If $30m - 2i_1 - 1 - mj \equiv 30m + 8(i - 1) + 1 - mj'$ and $i = 1$ or 2 then it can be checked that $1 \leq j' - j \leq 649$. In either case the triangle in $S_1$ or $S_2$ occurs after that of $S_0$ provided $j < 36m - 1913 \leq \min(j_0 - 649, n - j_0 - 826)$.

Now we turn to the proof of Theorem 3. The following lemma covers the cases when there are many 3 and 5-cycles.

**Lemma 26** There exist absolute constants $c_2$ and $c_3$ such that if $\sum_{m_i \in \{3,5\}} m_i \geq \frac{1}{3} \binom{n}{2} + c_2n$ and $\sum_{m_i \geq 72} \geq c_3n$ then the conclusion of Theorem 3 holds.

**Proof.** Write $n = 2n_0 + n_1$ with $n_0 = 72m + 1$ and $6 \leq n_1 \leq 149$. We shall take $c_3 = 1.4 \times 10^9$. Then $\binom{n}{2} \geq c_3n$, so $n > 2c_3 > 10^9$ and $m > 10^6$. Construct the self-avoiding trail of triangles in $K_n$ as before using Theorem 14. Let $r = \lceil \frac{n}{27} \rceil \leq 75$ and write the vertices of $K_n^{2}$ as $r$ disjoint pairs of vertices $P_i$, (or one singleton $P_1$ and $r - 1$ pairs $P_2, \ldots, P_r$ if $n$ is odd). Since $r < 10^5 < \frac{m-9}{6}$, we can construct $r$ disjoint collections $S_i$ of triangles using Lemma 24. Doubling up each vertex in $K_n$ gives us a trail of octahedra in $K_n^{2}$ as in the proof of Theorem 1. The collections $S_i$ are now collections of octahedra. We shall now modify the octahedra in the trail given by Theorem 14 so

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as to include all the additional edges.

First we deal with the edges in $K'_{n_1}$. Since $n_1 \geq 6$ we can decompose $K'_{n_1}$ into cycles of length at most $2n_1$. (For $n_1 = 6,7$ we can decompose $K'_{n_1}$ into triangles, for all larger $n_1$ we can decompose $K'_{n_1}$ into triangles and squares by [5].) Furthermore, we can ensure that at least one of these cycles is a triangle. For each such cycle $C_s$ pick a pair of non-adjacent vertices $u_1$ and $u_2$ in $K'_{2n_0}$.

If the cycle is $v_1, \ldots, v_s, v_{s+1} = v_1$ then we can construct $s$ paths $u_1v_1v_{i+1}u_2$ of length 3. Now pick a set of vertices $v'_1, \ldots, v'_s$ in $K'_{n_1}$ disjoint from the $v_i$ (possible since $s \leq \frac{n_1}{2}$). Now construct $s$ paths $u_1v'_iu_2$ of length 2. By Lemma 23, the vertex pair $u_1u_2$ must occur as non-link vertices of at least one out of any $c_1n_0$ consecutive octahedra. We can modify one such octahedron by adding one of the length 3 paths and one of the length 2 paths so that it is now isomorphic to $O_{[2,3]}$. Repeat this process with a different octahedron for each of the paths until we run out of paths. Both $u_1$ and $u_2$ are now joined to the same set of vertices $\{v_1, \ldots, v_s, v'_1, \ldots, v'_s\}$ in $K'_{n_1}$.

Now repeat this process with each of the other cycles that pack $K'_{n_1}$ in turn until we have used up all the edges of $K'_{n_1}$. We use a different pair $u_1u_2$ for each cycle. We still have a lot of choice as to which octahedra are modified this way. We shall choose these octahedra to be near the end of the sequence given by Theorem 14. To be more precise, the first such octahedron will be at least $n_0$ and at most $(1+c_1)n_0$ from the end of the sequence. Each successive octahedra above will be at least $n_0$ and at most $(1+c_1)n_0$ octahedra before the previous one. Since $m_1 \leq \lceil \frac{n-112}{20} \rceil < 2n_0$, the modified octahedra will be at sufficient distance from one another so that when cycles are packed into the trail of octahedra, each cycle will encounter at most one modified octahedron (and so the cycle will not accidentally meet itself in $K'_{n_1}$). Note that we have only modified some of the last $|E(K'_{n_1})|(1+c_1)n_0$ octahedra and we have also only involved at most $c' = \frac{1}{3}|E(K'_{n_1})|$ pairs $u_1u_2$. Later on, it may be necessary to pack a single triangle $C_{m_1} = C_3$ into $K'_{n_1}$. If this happens, pick one of the cycles above with $s = 3$ and replace the paths $u_1v_1v_{i+1}u_2$ with paths $u_1v_iu_2$. This changes three $O_{[2,3]}$’s into $O_{[2,3]}$’s and frees up the triangle $v_1v_2v_3$ in $K'_{n_1}$ without changing anything else.

Remove octahedra from $S_i$ which meet any of the pairs $u_1u_2$ used above. Since the octahedra in $S_i$ are vertex disjoint, this removes at most $c'$ octahedra from each $S_i$. Join $P_1$ to each of the remaining octahedra in $S_i$. If $n$ is odd and $i = 1$ then we also fill in the edges of the missing 1-factor of $O$ to obtain $K'_{7}$’s. These octahedra now become $K'_8$’s or $K'_7$’s and we have used most of the edges joining $K'_{2n_0}$ to $K'_{n_1}$ and most of the missing 1-factor $I$ of $K'_{2n_0} = K'_{n_1} - I$ when $n$ is odd. Each collection $S_i$ of octahedra can miss up to $n_0 - 3(24m - 144r - 143 - c') = 432r + 430 + 3c'$ pairs of vertices in $K'_{2n_0}$. All the other vertices are joined to $P_1$ and if $n$ is odd and $i = 1$ then all other pairs of vertices in $K'_{2n_0}$ are now joined to each other.

We now continue the algorithm above. If any edges remain then there must be an independent pair $u_1u_2$ of vertices in $K'_{2n_0}$ that have not been joined yet to all the vertices in $K'_{n_1}$. Both $u_1$ and $u_2$ are joined to the same set of vertices in $K'_{n_1}$, so there must be some pair $v_1, v_2$ in $K'_{n_1}$ (or just one $v_1$) which has not yet been joined to either $u_1$ or $u_2$. Find some octahedron between $n_0$ and $(1+c_1)n_0$ from the last octahedron modified at the beginning of this proof with the pair $u_1u_2$ as non-link vertices. Add two paths $u_1v_1u_2$ and $u_1v_2u_2$ (or the edge $u_1u_2$ if there is no $v_2$) to the octahedra to get $O_{[2,2]}$ (or $O_{[1,2]}$). Eventually we will have used up all the remaining edges. The trail of octahedra has been modified so that some of the last $4mn_0$ octahedra have been modified to $K'_8$’s or $K'_7$’s and some of the last $c'n_0$ octahedra have been modified to $O_{[a,b]}$ with $(a,b) = (1,2), (2,2)$ or $(2,3)$ and $c'' \leq (432r + 430 + 3c')r + [E(K'_{n_1})](1+c_1) \leq 1.62 \times 10^9$. In each case the modified octahedra are well separated—at least 6 apart for the $K'_8$’s and $K'_7$’s and at least $n_0$ apart for the $O_{[a,b]}$’s.
Pack the first $4mn_0 + s$ octahedra for some $0 \leq s \leq 10$ with $C_3$’s and $C_5$’s according to Lemma 22. We pack the $C_3$’s first, stopping when we have either packed more than $4mn_0$ octahedra, or if we run out of $C_3$’s. We also stop if we are less than six octahedra from the next $K_2$ or $K_3^*$ but do not have enough $C_3$’s to pack the next $K_2$ or $K_3^*$. By Lemma 22 we can stop packing $C_3$’s at any point in the sequence and so there will be at most 27 unpacked $C_3$’s left. Now pack the $C_5$’s using Lemma 22 until we have packed at least the first $4mn_0$ graphs in the sequence and at most 10 more $O$’s after these. This succeeds provided the total length of $C_3$’s and $C_5$’s is at least $12(4mn_0 + 10) + 4rn_0 + 27(3) \leq \frac{1}{2}(n_0^2) + c_2n$.

Pack the remaining cycles of length less that 72 using Lemma 18 and Lemma 16. As in the proof of Theorem 2, by using at most four cycles of length at least 72 we can use up all the cycles of length less than 72 and get a packing into some $G. \{a\}$ with $a \in A'$ where $G$ is some initial segment of our trail of modified octahedra. It is possible that we may pack a $\tilde{O}$ in Lemma 18. To avoid this, remove one triangle try again. In each of the exceptional cases in Lemma 18, the removal of a single triangle will make that case non-exceptional, so we no longer need to use a $\tilde{O}$. Pack this single triangle into $K_n'$, instead, modifying three $O_{[2,3]}$’s into $O_{[2,2]}$’s as described above. Provided the total length of cycles $m_i \geq 72$ is at least $4L + |E(O_{[2,3]})(\ell' + 1)n_0 \leq c_3n$ we will not encounter any of the modified octahedra that occur near the end of the sequence, and we will still have at least $n_0 > 7$ unmodified octahedra remaining before the first of these modified octahedra.

Now use Lemma 21 inductively to pack cycles of length at least 72 into the sequence of modified octahedra until we have reached the last modified octahedra. We now have a packing into some $G'. \{a'\}$ where $a' \in A'$ and $G$ is an initial segment of the sequence of octahedra that includes all the modified octahedra, but does not include the last $n_0$ octahedra in the sequence. Using at most four more cycles and Lemma 17 we can pack $G'', \{a''\}$ with $a'' \in A$. Now use Corollary 13 to complete the packing. In Corollary 13 we need the remaining cycles to be of length at least 40$L$. The remaining cycles are of length at least $12n_0 - \sum a' \geq 12n_0 - 4L - \sum a' \geq 12n_0 - 8L$. Hence we require $48L \leq 12n_0$. However, $48L \leq 48(\frac{2n_0 + 37}{20}) < 12n_0$ when $n_0 \geq 13$. As in Theorem 1, the cycles are packed properly in $K_n'$ since each cycle can meet at most $\lfloor \frac{m_i}{2} \rfloor$ octahedra and the trail of octahedra is $\lfloor \frac{n_0 + 18}{20} \rfloor$-self-avoiding. Also, the excursions into $K_n'$ are sufficiently far apart that no cycle meets itself in $K_n'$.

The result is now proved. A simple calculation shows that we can take $c_3 = 1.4 \times 10^9$ and $c_2 = 150$.

\[\square\]

**Lemma 27** There exist absolute constants $c_4$ and $c_5$ such that if $\sum_{m_i \in \{3,5\}} m_i \leq \frac{1}{2}(n_0^2) - c_4n$ and $\sum_{m_i \geq 72} m_i \geq c_5n$ then the conclusion of Theorem 3 holds.

**Proof**. Write $n = 2n_0 + n_1$ with $n_0 = 72m + 1$ and $6 \leq n_1 \leq 149$. We shall take $c_4 = 8.9 \times 10^7$. Since $\frac{1}{2}(n_0^2) \geq c_4n$, $n > 4c_4 > 10^8$ and $m > 10^3$. Construct the self-avoiding trail of triangles in $K_{n_0}$ as before using Theorem 14. Let $r = \lceil \frac{n_1}{2} \rceil \leq 75$ and write the vertices of $K_{n_1}'$ as $r$ disjoint pairs of vertices $P_i$, (or one singleton $P_1$ and $r - 1$ pairs $P_2, \ldots , P_r$ if $n$ is odd). Since $r < 10^4 < \frac{m+10}{4}$, we can construct sets $S_0, \ldots , S_r$ of triangles as in Lemma 25. Pick an integer $j$ so that $144m_j \geq \frac{1}{2}(n_0^2) - c_4n \geq \sum_{m_i \in \{3,5\}} m_i$. A simple calculation shows that we can take $j = 36m + 76 - c_4$. Translate all the vertices of all the triangles of $S_0, \ldots , S_r$ by $-mj$ so as to move the triangles along $12mj$ in the sequence of Theorem 14. Clearly properties 2 to 6 of Lemma 25 still hold and the triangles of $S_i$ all occur between the $(12mj + 1)^{st}$ and $(12mj + 6mn_0 + 211n_0)^{th}$ triangles of the sequence. Doubling up each vertex gives us a trail of octahedra as in Theorem 1.

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and the sets \( S_i \) are now sets of octahedra. We shall now modify the octahedra in the trail so as to include all the additional edges.

We deal with the edges in \( K'_n \) in exactly the same way as in Lemma 26. Once again we will have only modified some of the last \( |E(K'_n)|(1 + c_1)n_0 \) octahedra and have only involved at most \( c' = \frac{1}{3}|E(K'_n)| \) non-adjacent vertex pairs \( u_1u_2 \).

Remove octahedra from \( S_i \) which have these pairs as non-link vertices (when \( i > 0 \)) or link vertices (when \( i = 0 \)). By parts 4 and 5 of Lemma 25, this removes at most \( c' \) octahedra from each \( S_i \). Now for each \( i \geq 1 \) (if \( n \) even) or \( i \geq 3 \) (if \( n \) odd) join \( P_i \) to each of the non-link vertices of each octahedron in \( S_i \). These octahedra now become \( O\{2,2\}'s. We now pack most of the \( C_3 \)’s and \( C_5 \)’s into some of the first \( 12mj \) octahedra. By definition of \( j \) we will run out of \( C_3 \)’s and \( C_5 \)’s before the \((12mj + 1)\)st octahedron and so will not encounter any octahedron in any \( S_i \). Since four \( C_3 \)’s can be packed into \( O \) and twelve \( C_5 \)’s can be packed into \( O^5 \), the remaining \( C_3 \)’s and \( C_5 \)’s will have total length of at most \( 3(3) + 11(5) \). Now pack the other cycles of length less than 12 using Lemma 22. We first pack as many \( C_4 \)’s as possible. If \( n \) is odd, then each time we come to an octahedron in \( S_0 \), change the octahedron into the graph \( W \) by attaching the link vertices to the three vertices in \( P_1 \cup P_2 \) and joining the link vertices. Now use the \( C_4 \)’s to pack the subgraph \( W_{222} \). When two such graphs are packed, pack the remaining edges of the two \( W \)’s with three \( C_4 \)’s by pairing up the missing paths of length two from the first \( W \) with those of the second \( W \). We continue until we run out of \( C_4 \)’s. If we have an odd number of \( W \)’s, then “unmodify” the last one, converting it back to an \( O \). We also change back an octahedron if it occurred too early or late in the subset of packed condition. (In Lemma 22 we need a few octahedra at the start and end of the sequence to be unmodified.) We shall have at most two unpacked \( C_4 \)’s remaining (three would pack another \( O \)) and we may have changed back at most four octahedra. (One each at the start and end due to Lemma 22, one if we have an odd number of \( W \)’s and possibly one more since changing back octahedra gives us a few more \( C_4 \)’s to pack.) Now continue with \( C_5 \)’s in a similar manner. Once again, if \( n \) is odd we convert octahedra in \( S_0 \) into \( W \)’s. This time we pack one \( W_{222} \) and three \( W_{333} \)’s and use three \( C_5 \)’s to pack the remaining edges. (Make the missing paths of length 3 of the first \( W_{333} \) join \( v_1 \) and \( v_2 \), and those of the next join \( v_2 \) and \( v_3 \) and those of the last join \( v_3 \) and \( v_1 \). Now matching up two paths of length 3 and one of length 2 from distinct \( W \)’s gives three \( C_5 \)’s.) If we have some \( W_{333} \)’s or \( W_{222} \)’s left over or some modified octahedra occur too early or late in the packed sequence, then change these back into \( O \)’s. We shall have at most five \( C_5 \)’s remaining (six would pack another \( O^4 \)) and at most six octahedra changed back. Now continue with \( C_6 \)’s and \( C_7 \)’s packing \( W \)’s where possible. We shall have at most one \( C_6 \) and eleven \( C_7 \)’s remaining and we may need to change back at most three octahedra for the \( C_7 \)’s and five octahedra for the \( C_6 \)’s (up to two \( O\{2,2\} \) may need to be changed back into \( O \)’s to get the divisibility condition in Lemma 22 for the \( C_6 \)’s).

If \( n \) is odd, join \( P_1 \) and \( P_2 \) to the midvertices of octahedra in \( S_1 \) and \( S_2 \) respectively that have not been joined already when constructing \( W \)’s and which are at least \( 5m \) further on in the sequence than the last \( W \). When joining the singleton \( P_1 \) to the midvertices of an octahedron, join these vertices together as well so that the octahedron becomes \( O_{1,2} \). By part 6 of Lemma 25, all the link vertices of octahedra in \( S_0 \) not yet encountered will be joined to \( P_1 \cup P_2 \) at this point.

Now pack \( C_9 \)’s, \( C_{10} \)’s and \( C_{11} \)’s, packing \( O \)’s, \( O\{2,2\} \)’s and \( O_{1,2} \)’s as required. Once again, we may need to change back some \( O \)’s. A simple count shows that the total length of all cycles remaining of length less than 12 is now at most 393 and we have changed back at most 30 modified octahedra. Pack all these remaining cycles into the trail of octahedra together with at most four longer cycles using Lemma 18 for the cycles of length at most 9, and then Lemma 16 for the cycles of lengths

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10 and 11. If we encounter an $O_{[a,b]}$ we change it back to $O$ (this will occur at most three times since modified octahedra are separated by a distance of at least 12). If in Lemma 18 we need to use the graph $\tilde{O}$, then remove a triangle from the set of small cycles being packed and try again. Removing a triangle from the list of exceptional cases in Lemma 18 will never give another exceptional case, so we shall not need the $\tilde{O}$. The remaining triangle will be packed in $K'_{n_1}$ as in Lemma 26 by changing three $O_{[2,3]}$’s to $O_{[2,2]}$ near the end of the sequence.

Now pack cycles of lengths at least 12 in increasing order of length using Lemma 21 until we have passed the last octahedra in any $S_i$. We now have a packing into some graph $G_\{a\}$ with $a \in A'$ and $G$ is an initial segment of the sequence of octahedra.

We now estimate the number of edges remaining between $K_{2n_0}$ and $K_{n_1}$. We may have missed up to $c't$ squares and triangles avoiding vertex pairs $u_1u_2$ used at the beginning of the proof. We may also have missed at most $34 \times 4$ squares and triangles when we changed back some octahedra. Finally we may have up to 3827 missing squares and 3827 missing triangles because $S_0$ does not cover all of the vertices. We now continue the algorithm at the beginning of this proof. If any edges remain then there must be an independent pair $u_1u_2$ of vertices in $K_{2n_0}$ joined to a pair $v_1,v_2$ in $K'_{n_1}$ (or to just one $v_1$). Find some octahedron between $n_0$ and $(1+c_1)n_0$ from the last octahedron modified at the beginning of this proof with the pair $u_1u_2$ as non-link vertices. Add two paths $u_1v_1u_2$ and $u_1v_2u_2$ (or the edge $u_1u_2$ if there is no $v_2$) to the octahedra to get $O_{[2,2]}$ (or $O_{[1,2]}$). Eventually we will have used up all the remaining edges. The octahedra at the end of the sequence that have been modified all lie at most $c''n_0$ from the end where $c'' = (c't + 3827(2) + 34(4) + |E(K'_{n_1})|)/(1+c_1) \leq 1.45 \times 10^7$.

We now finish using the same argument as in Lemma 26. Provided the total length of cycles $m_i \geq 72$ is at least $4L + |E(O_{[2,3]})(c''+1)n_0 \leq c_5n$, and provided $12mj + 6mn_0 + 211n_0 + (c''+1)n_0 \leq 12mn_0$, we shall still have at least $n_0 > 7$ unmodified octahedra remaining before encountering the modified octahedra at the end of the sequence and after packing all the cycles of length less than 72. The argument of the end of Lemma 26 will finish the proof when these two inequalities hold. Note that the modified octahedra at the end of the sequence are at least $n_0 > \frac{L}{2}$ from any other modified octahedron, and the octahedra in any $S_i$ are at least $5m > \frac{L}{2}$ apart. Hence no cycle meets itself in $K'_{n_1}$. The octahedra in $S_i$ may be close to some in $S_j$, $j \neq i$, but this is not important since the modified octahedra in each $S_i$ will only meet $K'_{n_1}$ in the vertices of $P_i$ and the $P_i$ are disjoint. With $j = 36m + 76 - c_4$ as above, a simple calculation shows that we can take $c_4 = 8.9 \times 10^7$ and $c_5 = 1.3 \times 10^8$.

Finally we give the proof of Theorem 3.

Proof. of Theorem 3.

Corollary 20 proves the result when $n \geq N(71)$ and $\sum_{m_i \geq 72} m_i \leq \frac{9}{8} (\epsilon(71)n - 4)$. Lemma 26 proves the result when $\sum_{m_i \in [3,5]} m_i \geq \frac{1}{2} \binom{n}{3} + c_2n$ and $\sum_{m_i \geq 72} \geq c_3n$. Lemma 27 proves the result when $\sum_{m_i \in [3,5]} m_i \leq \frac{1}{2} \binom{n}{2} - c_4n$ and $\sum_{m_i \geq 72} \geq c_5n$. If we take $n$ sufficiently large so that $n \geq N(71)$, $\frac{1}{8} (\epsilon(71)n - 4) \geq \max(c_3, c_5)$ and $(c_2 + c_4)n \leq \frac{1}{6} \binom{n}{2}$ then in all cases at least one of these will prove the result. Indeed, we can take any $n \geq N_2 = \max(N(71), 1.2 \times 10^{10}/\epsilon(71))$. 

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6 Conclusion

It is possible to reduce the lower bound of \(72\) on \(m_i\) in Theorem 4 (and hence Theorem 1) substantially by using the packings of Lemma 15 in Lemma 11 and Lemma 12. However Theorem 4 is false without some restriction on the \(m_i\), since for example it is impossible to pack \(O^N\) with a \(C_8\) and \((3N-2)\ C_4\)’s for any value of \(N\). With considerably more effort, the following can however be proved.

**Theorem 28** There exists a constant \(c\) such that if \(m_1,\ldots,m_t\) are integers with \(3 \leq m_i \leq L, \sum_{i=1}^t m_i = 12N\) and \(\sum_{m_i \not\in\{3,4,7,8\}} m_i \geq cL\) then cycles \(C_{m_1},\ldots,C_{m_t}\) can be packed into \(O^N\).

Note that it is possible to avoid the use of \(\tilde{O}\) even when packing small cycles. All we need is that there are not too many \(C_3\)’s, \(C_4\)’s, \(C_7\)’s or \(C_8\)’s. Theorem 19 can be avoided, and hence the very large constants \(N_1\) and \(N_2\) in Theorem 2 and Theorem 3 can be reduced substantially if we avoid such cases.

The methods of Section 3 use cyclic Steiner Triple Systems. Theorem 14 is almost certainly not best possible, even for the Steiner triple system used in the proof. Similar arguments for other such systems (such as those constructed in [14]) should give a result similar to Theorem 14 for other values of \(n\), and hence should give results similar to Theorem 1 and Theorem 2 at least for \(n \equiv 2, 6 \mod 12\) with some linear upper bound on the \(m_i\)’s in terms of \(n\).

More generally, we need \(n \equiv 1, 3 \mod 6\) in Theorem 14 to have any Steiner Triple System. Also if a \(k\)-self-avoiding trail of triangles exists then and consecutive sequence of \(k\) triangles must involve \(2k + 1\) vertices, so \(2k + 1 \leq n\). We do however make the following conjecture.

**Conjecture 2** There exists an absolute constant \(c\) such that for all \(k\) and \(n\) with \(n \equiv 1, 3 \mod 6\) and \(n \geq 2k + c\) there exists a \(k\)-self-avoiding trail of triangles that pack \(K_n\).

Even if this conjecture were true, it would still only give an upper bound on \(m_i\) of \(\frac{n^2}{2} - c\) in all three theorems and still require \(n \equiv 2\) or \(6 \mod 12\) in Theorem 1 and Theorem 2. New ideas would still be needed to deal with cycles of length more than \(\frac{n^2}{2}\).

References


[12] A. Rosa, Alspach’s conjecture is true for \(n \leq 10\), *Mathematical reports*, McMaster University.


