ADJACENT VERTEX DISTINGUISHING EDGE-COLORINGS*

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Abstract. An adjacent vertex distinguishing edge-coloring of a simple graph \( G \) is a proper edge-coloring of \( G \) such that no pair of adjacent vertices meets the same set of colors. The minimum number of colors \( \chi'_a(G) \) required to give \( G \) an adjacent vertex distinguishing coloring is studied for graphs with no isolated edge. We prove \( \chi'_a(G) \leq 5 \) for such graphs with maximum degree \( \Delta(G) = 3 \) and prove \( \chi'_a(G) \leq \Delta(G) + 2 \) for bipartite graphs. These bounds are tight. For \( k \)-chromatic graphs \( G \) without isolated edges we prove a weaker result of the form \( \chi'_a(G) = \Delta(G) + O(\log k) \).

Key words. proper edge-colorings, chromatic number, bipartite graphs

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1. Introduction. Let \( G \) be a simple graph. We say a proper edge-coloring of \( G \) is adjacent vertex distinguishing, or an \( avd \)-coloring, if for any pair of adjacent vertices \( x \) and \( y \), the set of colors incident to \( x \) is not equal to the set of colors incident to \( y \). It is clear that an \( avd \)-coloring exists provided \( G \) contains no isolated edge. A \( k \)-\( avd \)-coloring is an \( avd \)-coloring using at most \( k \) colors. Let \( \chi'_a(G) \) be the minimum number of colors in an \( avd \)-coloring of \( G \). In [7] the following conjecture was made.

Conjecture 1. If \( G \) is a simple connected graph on at least \( 3 \) vertices and \( G \neq C_5 \) (a 5-cycle), then \( \Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2 \).

Since \( \chi'_a(G) \) is at least as large as the edge-chromatic number of \( G \) it is clear that \( \chi'_a(G) \geq \Delta(G) \), where \( \Delta(G) \) is the maximum degree of any vertex in \( G \). There are many examples of graphs for which \( \chi'_a(G) > \Delta(G) + 1 \). For example, consider a graph of the form \( G = K_{n,n} - H \), where \( H \) is a 2-factor of the complete bipartite graph \( K_{n,n} \) containing no \( C_4 \). Assume we have an \( avd \)-coloring of \( G \) using \( \Delta(G) + 1 \) colors. Then each vertex is not incident to precisely one color, and assigning this missing color to each vertex gives a proper vertex-coloring of \( G \) with \( \Delta(G) + 1 \) colors. Since \( G \) is bipartite with equal class sizes, the set of edges of a given color must miss the same number of vertices in each class. Hence each color occurs the same number of times on the vertices of each class. Since \( \Delta(G) + 1 = n - 1 \) there is a color that occurs at least twice in each class, but the vertices with this color do not form an independent set in \( G \). Hence \( \chi'_a(G) > \Delta(G) + 1 \).

More generally, if \( G \) is regular, then both \( \chi'_a(G) \) and the total chromatic number \( \chi_T(G) \) are at least \( \Delta + 1 \), and the above argument shows that \( \chi'_a(G) = \Delta + 1 \) if and only if \( \chi_T(G) = \Delta + 1 \). Hence any regular graph with \( \chi_T(G) > \Delta + 1 \) gives an example of a graph with \( \chi'_a(G) > \Delta + 1 \).

We shall prove the following upper bounds for \( \chi'_a(G) \).

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Theorem 1.1. If $G$ is a graph with no isolated edges and $\Delta(G) = 3$, then $\chi'_a(G) \leq 5$.

Theorem 1.2. If $G$ is a bipartite graph with no isolated edges, then $\chi'_a(G) \leq \Delta(G) + 2$.

Theorem 1.3. If $G$ is a $k$-chromatic graph with no isolated edges, then $\chi'_a(G) \leq \Delta(G) + O(\log k)$.

In particular, Conjecture 1 holds for all bipartite graphs and all graphs with $\Delta(G) \leq 3$. Note that even for bipartite graphs, Conjecture 1 is best possible, as the example above shows. Theorem 1.3 is not best possible; indeed, Hatami [5] has recently shown using probabilistic methods that $\chi'_a(G) \leq \Delta(G) + 300$ for sufficiently large $\Delta(G)$, which is stronger than Theorem 1.3 for graphs with an extremely high chromatic number. Theorem 1.1 will be proved in section 2, Theorem 1.2 will be proved in section 3, and Theorem 1.3 will be proved in section 4.

Adjacent vertex distinguishing colorings are related to vertex distinguishing colorings in which every pair of vertices sees distinct color sets. This concept has been studied in many papers; see, for example, [1, 2, 3, 4, 5, 6].

2. Graphs with $\Delta(G) = 3$. We start with the special case of regular graphs having a Hamiltonian cycle. Our coloring scheme is based on the idea of using the four elements of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to color the Hamiltonian cycle, defining the colors used algebraically, and a new fifth color for the chords forming a 1-factor. Local adjustments will be made to complete the colorings.

Lemma 2.1. If $G$ is a 3-regular Hamiltonian graph, then $G$ has a 5-avd-coloring.

Proof. Let the five colors be the elements $\{0, a, b, c\}$ of the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$ together with the extra color 5. We have a commutative and associative addition defined on $K$ such that $x + x = 0$ for all $x$ and $a + b = c$. Let $C = x_1 \ldots x_n$ be a Hamiltonian cycle of $G$ and let $I$ be the remaining 1-factor of $G$. We may assume $G \neq K_4$ (see Figure 1 for a 5-avd-coloring of $K_4$), so by Brooks’ theorem, $G$ has a vertex 3-coloring $f : V(G) \rightarrow \{a, b, c\}$. We may also assume that each of the three colors occurs at least once on $G$; otherwise a single vertex can be recolored to introduce the missing color. Let $S = \sum_{i=1}^{n} f(x_i) \in K$.

If $S = 0$, then label $x_n x_1$ with 0 and inductively label $x_i x_{i+1}$ for $i = 1, \ldots, n - 1$ so that $f(x_i)$ is the sum (in the group $K$) of the colors on $x_{i-1} x_i$ and $x_i x_{i+1}$. Equivalently, the color on $x_i x_{i+1}$ is the sum of the color on $x_{i-1} x_i$ and $f(x_i)$. Then $f(x_n)$ is the sum of the colors on $x_n x_1$ and $x_n x_1$. Color the 1-factor $I$ with color 5. Each vertex $x$ sees color 5 and two colors from $K$ summing to $f(x)$. Since $f(x) \neq 0$ these two colors from $K$ are distinct, and since $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$, the color sets at $x$ and $y$ must be distinct. Thus the coloring is a 5-avd-coloring of $G$.

Now suppose $S \neq 0$. Without loss of generality we may assume $S = c$. Pick any vertex $x_i$ with $f(x_i) = c$. Let $x_i x_j \in I$. Then $f(x_j)$ is either $a$ or $b$. Recolor $x_j$ with $b$ or $a$, respectively. Now $S = 0$ and we can recolor the edges of the Hamiltonian cycle as above (see Figure 1). Coloring $I$ with 5 gives a proper edge-coloring that distinguishes adjacent vertices, except possibly at $x_j$. Since $f(x_i) \neq f(x_j)$ the pair of colors from $K$ meeting $x_i$ cannot be disjoint from the pair meeting $x_j$. Hence there must be some color of $K$ missing from the edges incident to $x_i$ or $x_j$. Recoloring the edge $x_i x_j$ with this missing color gives a 5-avd-coloring of $G$. The vertices $x_i$ and $x_j$ are distinguished from each other since $f(x_i) \neq f(x_j)$ and are distinguished from all other vertices since all other vertices meet color 5.

We shall now assume that $G$ is 3-regular with a 1-factor, but is not necessarily
Hamiltonian. Since \( G \) has a 1-factor, \( G \) can be written as a union of this 1-factor and a collection of cycles. We shall show that under certain conditions we can extend a partial coloring to each cycle in turn.

We shall first find suitable colorings of graphs \( H \) of the following form. Let \( H \) be a cycle \( C = x_1 \ldots x_n \) with some extra 3-stars and chords added. To be precise, partition \( V(C) = V_I \cup V_C \cup V_S \). For \( x_i \in V_I \), \( H \) will contain an edge \( x_ix'_i, x'_i \notin V(C) \), where \( x'_i \) is joined to two degree 1 vertices. For \( x_i \in V_C \), \( H \) will contain a chord \( x_ix_j \), where \( x_j \in V_C \). For \( x_i \in V_S \), \( d_H(x_i) = 2 \) (see Figure 2).

We shall color such graphs so that adjacent degree 3 vertices are distinguished. We shall specify the colors incident to the \( x'_i \) for all \( x_i \in V_I \), and try to extend the coloring to the rest of \( H \).

**Lemma 2.2.** Let \( H \) be a graph as above with \( |V_S| \geq 2 \). Suppose the edges incident to each \( x'_i \) with \( x_i \in V_I \) are properly colored with colors from \( K \cup \{5\} \) and \( x_ix'_i \) is colored 5. Then we can properly color the remaining edges of \( H \) with colors from \( K \cup \{5\} \) so that adjacent degree 3 vertices are distinguished. Moreover, if \( x_i \in V_S \), then we can ensure that either \( x_i \) meets color 5 or both neighbors of \( x_i \) meet color 5.

**Proof.** We partition \( V_S \) into two sets \( V_I \) and \( V_M \) as follows. If \( x, y \in V_S \) are adjacent on \( C \), color the edge \( xy \) with color 5 and place \( x \) and \( y \) in the set \( V_M \). Repeat with other adjacent pairs of \( V_S \) (that have not been used already) until \( V_I = V_S \setminus V_M \) is an independent set. We shall now 3-color the degree 3 vertices of \( H \) with \( \{a,b,c\} \). For \( x_i \in V_I \), set the color of \( x'_i \) to be the sum of the two colors of \( K \) incident to \( x'_i \). Extend this vertex-coloring to a proper vertex-coloring of \( V(H) \setminus V_S \) using a greedy algorithm—proceed around \( C \), starting at any vertex immediately after a vertex of \( V_S \), coloring each vertex of \( V_Y \cup V_C \) in turn with any color from \( \{a,b,c\} \) that ensures that the coloring is still proper.

![Figure 1. Colorings in Lemma 2.1. Dotted edges are colored 5.](image1)

![Figure 2. Graph H with V_I = \{x_1, x_2, x_6\}, V_C = \{x_5, x_8\}, V_S = \{x_3, x_4, x_7\}.](image2)
If \( V_M = \emptyset \), then \(|V_I| \geq 2\). By coloring vertices of \( V_I \) (not necessarily properly) with colors from \( \{a, b, c\} \), we can ensure that the sum of the vertex colors on \( C \) is \( 0 \in K \). If \( V_M \neq \emptyset \), color \( V_I \) arbitrarily with \( \{a, b, c\} \). Color the uncolored edges around \( C \) as in Lemma 2.1. At each vertex we add the vertex color in the Klein group to get the color of the next edge. The edge after any pair of vertices from \( V_M \) can be colored arbitrarily with any color from \( K \). Color each chord of \( C \) with color 5. The resulting coloring satisfies the conditions of the lemma.

Note that if we add an edge \( x_i x'_i \) to \( H \) for some \( x_i \in V_S \), then we can color \( x_i x'_i \) with some color from \( K \) so that the new coloring is still proper and distinguishes degree 3 vertices. Indeed, if \( x_i \) meets color 5 in a coloring given by Lemma 2.2, then there are three colors which make the coloring proper and at most two of these will fail to distinguish \( x_i \) from \( x_{i+1} \) or \( x_{i-1} \). If \( x_i \) does not meet color 5, then \( x_i x'_i \) may be colored with either of the remaining colors of \( K \) since both \( x_{i+1} \) and \( x_{i-1} \) meet color 5.

**Lemma 2.3.** Let \( H \) be a graph as above with \( V_S = \emptyset \) and \( x_1 \in V_Y \). Suppose the edges incident to each \( x_i \) with \( x_i \in V_Y \) are properly colored with colors from \( K \cup \{5\} \), \( x_i x'_i \) is colored 5, and either of the following two conditions holds:

(a) All the edges incident to \( x'_i \) are colored, and one of the two edges that are incident to \( x'_i \) but not \( x_i \) is colored 5.

(b) The edges incident to \( x'_i \) are colored, except for \( x_1 x'_1 \) which remains uncolored.

Then the coloring can be completed to form a 5-avd-coloring of \( H \). Moreover, in this coloring, \( x_1 x'_1 \) is not colored 5, but either \( x_1 \) meets color 5, or both \( x_2 \) and \( x_n \) meet color 5.

**Proof.** We shall provisionally color all chords \( x_i x_j \) of \( C \) with color 5. As in the proof of Lemma 2.1 we shall 3-color the vertices of \( H \) with \( \{a, b, c\} \). Each \( x_i \) for \( x_i \in V_Y \) is assigned the sum of the colors of \( K \) meeting it in \( H \). We 3-color the vertices \( x_2, \ldots, x_n \) in turn so that the coloring is proper using a greedy algorithm. The vertex \( x_1 \) will remain uncolored. Let this coloring be denoted by \( f \) and write \( S = \sum_{i=2}^{n} f(x_i) \). If \( S \neq 0 \), then assign \( x_1 x_2 \) any color of \( K \), and color the edges around the cycle as in the proof of Lemma 2.1. This gives four possible avd-colorings of \( H - x'_1 \), depending on the choice of color for \( x_1 x_2 \), and yields either \( \{0, S\} \) or \( K \setminus \{0, S\} \) as the pair of colors on \( x_n x_1 \) and \( x_1 x_2 \).

Assume that there is a chord \( x_i x_j \) of \( C \) which does not meet either \( x_2 \) or \( x_n \). Suppose without loss of generality that \( f(x_i) = a \) and \( f(x_j) = b \). Recolor either \( x_i \) or \( x_j \) with \( c \) and change the color of \( x_i x_j \) to some color of \( K \) as in the proof of Lemma 2.1 so as to keep the coloring proper. Note that the coloring distinguishes \( x_i \) and \( x_j \) from all their neighbors, since their neighbors all meet color 5. In this way we can construct colorings with three distinct values of \( S \) (the original coloring, the coloring changing \( f(x_i) \), and the coloring changing \( f(x_j) \)). At least two of these will have \( S \neq 0 \), and by varying the choice of color on \( x_1 x_2 \) as above, we obtain colorings with four possible values for the pair of colors on \( x_n x_1 \) and \( x_1 x_2 \). These four pairs form the edges of a \( C_4 \) inside \( K_K \)—the complete graph on the color set \( K \). Moreover, both \( x_2 \) and \( x_n \) meet color 5, so are distinguished from \( x_1 \) regardless of the color (in \( K \)) of \( x_1 x'_1 \). In case (a) we are done since we can choose a coloring for which the pair of colors on \( x_1 x_2 \) and \( x_1 x_2 \) avoids the color of \( x_1 x'_1 \). In case (b) we are done since we can choose a coloring for which the pair of colors on \( x_n x_1 \) and \( x_1 x_2 \) is neither equal nor disjoint from the pair that meet \( x'_1 \). Then there is at least one remaining color of \( K \) with which to color \( x_1 x'_1 \).

Assume now that \( x_2 x_j \) is a chord of \( C \). In case (b) color \( x_1 x'_1 \) arbitrarily with
some color of \( K \) so that the coloring is proper at \( x_1 \). Restart from scratch and 3-color the vertices of \( H \) with \( \{a, b, c\} \) as follows. As before, \( x_1 \) gets the sum of the colors of \( K \) meeting it when \( x_1 \in V_y \). For the cycle \( C \), we start the coloring at \( x_{j-1} \), working backwards greedily until we reach \( x_2 \). The vertex \( x_2 \) can be colored in two ways. We pick one that ensures that \( S' = \sum_{i=2}^{j-1} f(x_i) \neq 0 \). Now continue coloring greedily with \( x_1 \), and then \( x_n, \ldots, x_{j+1} \). The vertex \( x_j \) will remain uncolored.

Starting at \( x_1 \) and working backwards around \( C \), color the edges so that \( f(x_i) \) is the sum of the colors of \( K \) meeting \( x_i \). For the edge \( x_j x_{j-1} \) pick either color of \( K \) that is not the same as the color of \( x_{j+1} x_j \), or the sum of this color and \( S' \). We continue coloring the edges of \( C \) as in Lemma 2.1 until we get to \( x_2 \). Color \( x_1 x_2 \) with color 5 and color \( x_2 x_j \) with the sum of \( f(x_2) \) and the color on \( x_2 x_3 \) (see Figure 3). This color will be the sum of \( S' \) and the color on \( x_j x_{j-1} \), so it is distinct from the colors on \( x_{j+1} x_j \) and \( x_j x_{j-1} \). The resulting coloring satisfies the conditions of the lemma.

A similar argument deals with the case when there is a chord of the form \( x_n x_j \), so we may now assume there are no chords, \( V_C = \emptyset \). Restart by coloring the vertices of \( C - x_1 \) with \( \{a, b, c\} \) as follows. Assume \( f(x'_2) = \cdots = f(x'_j) = a \neq f(x'_{j+1}) \) (or \( j = n \)). Color \( x_{j+1} \) with \( a \) and greedily color \( x_i \) for \( i > j + 1 \). The vertices \( x_3, \ldots, x_j \) can be colored alternately by \( b \) and \( c \), starting with either \( b \) or \( c \). The vertex \( x_2 \) will be colored \( a, b, \) or \( c \) (possibly equal to the color of \( x'_2 \), but not equal to the color of \( x_3 \)). Let \( S = \sum_{i=2}^{n} f(x_i) \).

We now list the possible colorings. For each choice of colorings of \( x_2 \) and \( x_3 \), there are four possible values of \( S \) depending on the value of \( S' = \sum_{i=j+1}^{n} f(x_i) \) and \( j \). Table 1 lists the various possibilities. Each value of \( S' \) and \( j \) has a column in the table for \( S \). Since \( f(x_2) \) and \( f(x_3) \) can be changed independently of \( S' \) we have several choices for the vertex-coloring for each \( S' \) and \( j \). We describe several cases in which we can find a suitable corresponding edge-coloring.

**Case A.** \( f(x_2) \neq f(x'_2) \), \( S \neq 0 \).
As in Lemma 2.1, we can edge color $C$ starting at $x_1x_2$. Since $S \neq 0$, the colors on $x_1x_n$ and $x_1x_2$ will be distinct. Depending on the choice of $x_1x_2$, the pair of colors meeting $x_1$ can be chosen to be either $\{0, S\}$ or $K \setminus \{0, S\}$. (We assume $x_1x'_1$ is uncolored for now.)

**Case B.** $f(x_2) = f(x'_2) \neq S, S \neq 0$.

As before we color the edges of $C$. However, this time only two choices for $x_1x_2$ are allowed since we must ensure that $x_2$ is distinguished from $x'_2$ (either color not meeting $x'_2$ will do for $x_1x_2$). These choices differ by the addition of $f(x_2)$ to every edge of $C$, and since $f(x_2) \notin \{0, S\}$, this swaps the pairs of colors $\{0, S\}$ and $K \setminus \{0, S\}$ on $x_nx_1$ and $x_1x_2$.

**Case C.** $f(x_2) = f(x'_2) = S, S \neq 0$.

Unfortunately, both choices above of the color for $x_1x_2$ give the same pair of colors on $x_nx_1$ and $x_1x_2$. Hence we can only guarantee that colorings exist making $x_1$ meet one of the pairs $\{0, S\}$ or $K \setminus \{0, S\}$.

For each $S'$ and $j$ (corresponding to a column in Table 1) there are always at least two possible nonzero values for $S$. Moreover, for any choice of $f(x'_2)$, we can find two choices of $f(x_2)$ and $f(x_3)$ with distinct values of $S \neq 0$, at least one of which has either $f(x_2) \neq f(x'_2)$ or $f(x_2) = f(x'_2) \neq S$ (the second case occurs only in the column indicated by *). Hence the set of pairs of colors meeting $x_1$ can be chosen as any edge of a path of edge length 3 in $K_K$ (one value of $S$ gives a matching in $K_K$, the other value gives at least one more edge in $K_K$).

In case (a) we are now done, since there is always a choice of the pair of colors that does not include the color on $x_1x'_1$. Also, $x_1$ and $x'_1$ are distinguished since only $x'_1$ meets color 5. In case (b) there is some choice for this pair of colors that is not equal or disjoint from the pair of colors meeting $x'_1$. Hence there is a choice of color in $K$ for $x_1x'_1$ which makes the coloring proper and distinguishes $x_1$ and $x'_1$. $\square$

**Theorem 2.4.** If $G$ is a 3-regular graph containing a 1-factor, then there exists a 5-avd-coloring of $G$.

**Proof.** Without loss of generality we may assume $G$ is connected. Decompose $G$ as a 1-factor $I$ and a union of cycles $C_i$. If there is only one cycle, then $G$ is Hamiltonian and we are done by Lemma 2.1. Otherwise construct a new graph $M$ with vertex set $V(M)$ equal to the set of cycles $C_i$ and edges joining $C_i$ and $C_j$ when there is an edge of $I$ joining some vertex of $C_i$ to some vertex of $C_j$. Since $G$ is connected, $M$ is also connected. Pick a spanning tree $T$ of $M$. Decompose $T$ as a vertex disjoint union of stars $S_j$, $|V(S_j)| \geq 2$. For each $S_j$ let $G_j$ be the subgraph of $G$ with an edge set made up of the edges of the cycles $C_i$ of $S_j$, together with their chords in $G$ and one edge of $I$ joining $C_i$ and $C_i'$ for each edge $C_iC_i'$ of $S_j$. Color $G$ in the following way. Each edge (of $I$) that does not lie in any $G_j$ will be colored 5. Now color each $G_j$ in turn. If the star $S_j$ has at least 3 vertices in $M$, use Lemma 2.2 to color the central cycle $C_{i_0}$ of $S_j$. The graph $H$ of Lemma 2.2 consists of $C_{i_0}$, its chords in $G$, and some 3-stars. The vertices of $C_{i_0}$ incident to an edge joining $C_{i_0}$ to another cycle in $G_j$ will be placed in $V_S$, and we attach a 3-star to each remaining vertex of $C_{i_0}$ that does not meet a chord of $C_{i_0}$. The edges of this 3-star correspond in an obvious way to some of the edges of $G$ (although the degree 1 vertices and the edges incident with degree 1 vertices of $H$ may not necessarily be distinct in $G$). We color the edges of the 3-stars with the corresponding colors already assigned in $G$, or arbitrarily (but properly) if no color has been assigned yet. Note that the edge of a 3-star incident to $C_{i_0}$ will be colored 5. Lemma 2.2 now extends the coloring to the edges and chords of $C_{i_0}$. Now color the edges $x_iy_i$ joining $C_{i_0}$ to the other cycles $C_i$
of \( G_j \) with some color of \( K \) if \( x_1 \) meets color 5 on \( C \) in such a way that the coloring is avd on \( C \) (see note after Lemma 2.2). Otherwise leave \( x_iy_i \) uncolored. We now color the other cycles \( C_i \) of \( G_j \) using Lemma 2.3 in a similar manner using the edge \( x_iy_i \) as the edge \( x_1x_1' \) of Lemma 2.3. The conditions of Lemma 2.3 ensure that we can find a coloring that is a 5-avd-coloring regardless of the choices of colors on the edges already colored. If the star \( S_j \) consists of just two vertices, use Lemma 2.3 on both constituent cycles. For the first cycle we use case (b) of Lemma 2.3. This will result in the edge \( x_1x_1' \) between the cycles being colored. If \( x_1 \) does not meet color 5, then uncolor \( x_1x_1' \). Now color the other cycle using case (a) or (b) of Lemma 2.3. If \( x_1x_1' \) is recolored with a new color, then \( x_1 \) does not meet color 5, but both its neighbors on the first cycle do. Hence the coloring is still avd on the first cycle. \( \square \)

**Proof of Theorem 1.1.** We shall prove Theorem 1.1 by induction on \(|E(G)|\). Paths and cycles on at least 3 vertices have 5-avd-colorings [7], so we may assume that \( G \) is connected with maximum degree 3.

Assume \( x \) is a vertex of degree 1 in \( G \). Let \( y \) be the neighbor of \( x \). Then \( y \) is of degree 2 or 3. Since \( G \neq P_3 \) we can find a 5-avd-coloring of \( G' = G - x \) by induction. In \( G' \), \( y \) has degree at most 2, so there are at least three colors not incident to \( y \). At most two of these colors cannot be used to color \( xy \), as they may result in \( y \) meeting the same set of colors as some neighbor in \( G' \). However, there is still at least one color that can be given to \( xy \) so that the coloring is avd. Hence we may assume \( G \) contains no degree 1 vertex.

Assume two vertices of degree 2 are adjacent in \( G \). Let \( x_0x_1x_2\ldots x_n \), \( n > 2 \), be a suspended trail in \( G \), i.e., a trail with \( d_G(x_0) = d_G(x_n) = 3 \) and \( d_G(x_i) = 2 \) for \( 0 < i < n \). If \( x_0 \neq x_n \), let \( G' \) be the graph obtained by contracting this path to \( x_0y_0x_n \). If \( x_0 = x_n \), let \( G' \) be the graph obtained by deleting the vertices \( x_1, \ldots, x_{n-1} \) and adding two degree 1 vertices \( y, z \) to \( x_0 = x_n \) (see Figure 4). By induction \( G' \) has a 5-avd-coloring. We may assume without loss of generality that the edge \( x_0y \) has color 1 and \( x_iy \) (or \( x_nz \)) has color 2. The edges \( x_ix_{i+1} \) of \( G \) can be colored with 1 for \( i = 0, 2 \) for \( i = n - 1 \), and cyclically with the colors \( \{3, 4, 5\} \) for other values of \( i \).

Hence we can assume that any vertex of degree 2 is adjacent only to vertices of degree 3. If \( G \) contains a bridge \( xy \), let \( G_1 \) and \( G_2 \) be components of \( G - xy \) with \( x \in V(G_1) \) and \( y \in V(G_2) \). Give \( G_1 \cup xy \) and \( G_2 \cup xy \) 5-avd-colorings by induction. (These graphs have smaller edge counts than \( G \) since \( G \) has no degree 1 vertices.) By permuting the colors on \( G_2 \cup xy \), we can assume the edge \( xy \) receives the same color in each coloring and the color set incident to \( x \) in \( G_1 \cup xy \) is not the same as the color set incident to \( y \) in \( G_2 \cup xy \). This now gives a 5-avd-coloring of \( G \) (see Figure 5).

Hence we can assume that \( G \) is a graph with maximum degree 3, no vertices of degree 1, no pair of adjacent degree 2 vertices, and is bridgeless. By Tutte’s 1-factor theorem, any cubic graph without a 1-factor must contain at least three bridges, so if \( G \) contains no degree 2 vertices, we are done by Theorem 2.4. If \( G \) contains degree 2 vertices, then let \( G' \) be the graph obtained by taking two copies of \( G \) and joining their corresponding degree 2 vertices by an edge. Then \( G' \) is 3-regular and contains at most

![Figure 4. The case when G contains adjacent degree 2 vertices.](image-url)
Fig. 5. The case when $G$ contains a bridge.

one bridge. Hence $G'$ has a 1-factor and so by Theorem 2.4 $G'$ has a 5-avd-coloring. This coloring of $G'$ induces a 5-avd-coloring of $G$ since no two vertices of degree 2 are adjacent in $G$. □

3. Bipartite graphs. If $G$ has an edge-coloring with colors $c_1, \ldots, c_k$, write $G\{c_1, \ldots, c_r\}$ for the subgraph of $G$ consisting of all the vertices of $G$ together with the edges of $G$ that are colored with a color in $\{c_1, \ldots, c_r\}$. Write $S(v)$ for the set of colors incident to $v$ and $\chi' = \chi'(G)$ for the edge-chromatic number of $G$.

The bound $\chi'_a(G) \leq \Delta(G) + 3$ for regular bipartite graphs comes rather easily using the 1-factorization of regular bipartite graphs. To see this, observe that a 2-regular bipartite graph $H$ with bipartition $V(H) = A \cup B$ has a straightforward 5-avd-coloring along each cycle such that $S(a) \in \{(1,2), (3,4), (3,5)\}$ and $S(b) \in \{(1,4), (2,3), (4,5), (1,3)\}$ for every $a \in A$ and $b \in B$. For $\Delta(G) > 2$ use this coloring for a 2-factor $H \subseteq G$ and give $G \setminus H$ any proper coloring with the remaining $\Delta(G) - 2$ colors. To obtain the bound $\chi'_a(G) \leq \Delta(G) + 2$ for any bipartite graph, however, much more effort will be required.

**Lemma 3.1.** If $G$ is a bipartite graph with no isolated edges, then there exists a proper edge-coloring with colors $\{1, \ldots, \chi'(G)\}$ such that

A. if $uv \in E(G) \setminus E(G\{1,2\})$, then either $\{1,2\} \subseteq S(u)$ or $\{1,2\} \subseteq S(v)$;

B. if $C$ is a cycle in $G\{1,2\}$ which does not meet color 3 in $G$, then $\{1,2,3\} \subseteq S(v)$ for every neighbor $v$ in $G \setminus C$ of any vertex of $C$;

C. if $C$ is a cycle in $G\{1,2\}$ which does meet color 3 in $G$, then there exists a $u \in V(C)$ and $uv \in E(G\{3\})$ with $\{1,2\} \subseteq S(v)$ (we allow $v \in V(C)$); and

D. if $uv$ is an isolated edge in $G\{1,2,3\}$, then $S(u) \neq S(v)$.

**Proof.** Consider the set of edge-colorings of $G$ with $\chi'$ colors. For all such colorings pick one such that

1. $G\{1,2\}$ has maximal edge count;

2. subject to (1), $G\{1,2\}$ has the minimum number of components (counting isolated vertices as components);

3. subject to (1)–(2), $G\{3\}$ has maximal edge count; and

4. subject to (1)–(3), the number of edges $uv$ in $G$ failing condition D is minimal.

We shall show that such a coloring satisfies conditions A–D.

**Condition A.** Let $uv \in E(G) \setminus E(G\{1,2\})$ be an edge with $\{1,2\} \nsubseteq S(u), S(v)$. Then $u$ and $v$ are either isolated vertices or the end-vertices of paths in $G\{1,2\}$. By recoloring $uv$ with either color 1 or 2 (and possibly interchanging colors 1 and 2 in the component of $v$ in $G\{1,2\}$) we obtain a proper edge-coloring with more edges colored $\{1,2\}$, contradicting (1). Note that if $u$ and $v$ are end-vertices of the same path in $G\{1,2\}$, then since $G$ is bipartite, the edge $uv$ can be recolored without changing any colors on this path.

**Condition B.** Assume $uv \in E(G)$ with $u \in V(C)$, $v \notin V(C)$, and $\{1,2,3\} \nsubseteq S(v)$. Note that $uv$ is not colored with 1, 2, or 3. If $3 \notin S(v)$, then we can recolor $uv$ with 3, contradicting (3). Hence without loss of generality $1 \notin S(v)$. Recolor $uv$ with 1 and
The paths in $G_2$ on the path from $z$ is either an isolated vertex or the end of a path in $G\{1,2\}$. Recolor $uv$ and $wz$ with 1, $uw$ with 3, and, if necessary, swap colors 1 and 2 on the path from $z$ in $G\{1,2\}$ so as to make the coloring proper (see Figure 6). If the paths in $G\{1,2\}$ meeting $v$ and $z$ the same, then recoloring this path will be unnecessary since $G$ is bipartite. We now have a new coloring with more edges in $G\{1,2\}$, contradicting (1).

Condition D. Let $u_1v_1$ be an isolated edge of $G\{1,2,3\}$. By Condition A, $u_1v_1$ is colored with either 1 or 2. Since $G$ contains no isolated edge, we can assume that $d_G(u_1) \geq 2$ and that $u_1$ meets another color $k > 3$ on some edge of $G$. Swap colors 3 and $k$ along a Kempe chain (component path of $G\{3,k\}$) starting at $u_1$ in $G$. By condition (3) we may assume that the last edge of this chain is recolored $k$. This reduces the number of edges failing condition D unless after the recoloring the other end-vertex $v_2$ of this chain lies in some isolated edge $u_2v_2$ of $G\{1,2,3\}$ and $S(u_2) = S(v_2)$. In this case $u_2$ also meets color $k$, so we can form a new Kempe chain starting at $u_2$ using colors 3 and $k$, disjoint from the $u_1$-$v_2$ chain. Repeating this process we get a sequence of Kempe chains on colors 3 and $k$ from $u_i$ to $v_{i+1}$. Note that properties (1)–(3) still hold after these recolorings. Eventually this process must terminate with a coloring reducing the number of edges failing condition D. Note that all the Kempe chains are vertex disjoint, and none end at $v_1$ since otherwise some recoloring would increase the number of edges colored 3, contradicting (3).

Proof of Theorem 1.2. Color $G$ as in Lemma 3.1. We shall recolor the edges of $G\{1,2,3\}$ with the five colors from $K \cup \{3\}$, where $K = \{0, a, b, c\}$ is the Klein group. This will give an avd-coloring with $\chi'(G) + 2 = \Delta(G) + 2$ colors. In addition, a vertex $v$ will meet color 3 in the new coloring only if it met 3 in the original coloring, so $|S(v) \cap K|$ will be at least as large as the original degree of $v$ in $G\{1,2\}$.

The edges of $G\{1,2\}$ form a set of vertex disjoint paths and even cycles. Construct a new graph $M$ with a vertex set equal to the nonsingleton components $C_i$ of $G\{1,2\}$ and edges joining $C_i$ and $C_j$ when either

1. there is an edge of $G\{3\}$ joining a vertex of degree 2 in $C_i$ to a vertex of degree 2 in $C_j$; or
2. either $C_i$ or $C_j$ is a single edge and there is an edge of $G\{3\}$ joining any vertex of $C_i$ to any vertex of $C_j$.

As in the proof of Theorem 2.4, we take a star decomposition $\{S_j\}$ of a spanning forest of $M$ and consider a corresponding subgraph $G'$ of $G\{1,2,3\}$ in $G$ consisting of the induced subgraphs in $G\{1,2,3\}$ of each cycle $C_i$ and a choice of edges from $G\{3\}$.
as above, joining $C_i$ and $C_j$ when $C_iC_j$ is an edge of one of the stars in the star decomposition. Note that the graph $M$ may contain isolated vertices, so some of the stars may be isolated vertices as well. We shall color every edge that does not lie in $G'$ with its original color in $G$. The colors 1 and 2 do not appear on these edges. The subgraph $G'$ will be colored with colors from $K \cup \{3\}$ so as to obtain an avd-coloring of $G$ using at most two more colors.

We say a component $C_i$ of $G\{1,2\}$ is bad if it is either a single edge where the end-vertices are not distinguished in the coloring of $G$, or a cycle of length congruent to 2 mod 4 that meets color 3, but has no color 3 chord. All other $C_i$'s will be called good.

If $C_i$ is a bad cycle, then by condition C, $C_i$ is adjacent by an edge of $G\{3\}$ to a vertex of degree 2 in $G\{1,2\}$. In particular, $C_i$ is not isolated in $M$. If $C_i$ is a bad edge, then by condition D it meets an edge of $G\{3\}$, and so once again $C_i$ is not isolated. Thus all isolated components are good.

Now we consider the stars $S_j$. Suppose we have a star with central component $C_0$ and end-components $C_1, \ldots, C_r$. If $r \geq 2$, delete the edge of $G'$ from $C_0$ to good components $C_i$ in $S_j$, $i > 0$, until either $r = 1$ or all $C_i$, $i > 0$, are bad. If $r = 1$ and $C_0$ and $C_i$ are good, delete the edge joining them in $S_j$. If $C_0$ is bad and $C_i$ is good, we consider $C_i$ to be the center of the star. Furthermore, if $C_0$ is an edge, then $C_i$ is not an edge (otherwise we would have two adjacent vertices of degree 1 in $G\{1,2\}$, contradicting condition A). In this case also we swap $C_0$ and $C_i$, so we can assume without loss of generality that $C_0$ is not a single edge when $r = 1$ (or $r > 2$). Any edge deleted from $G'$ will remain colored 3 in our final coloring.

Hence we may assume each star $S_j$ is either an isolated good $C_0$ or a star with all end-components either bad or single edges. Also, the color 3 edges in $G'$ joining $C_0$ to the end-components are incident to degree 2 vertices of $C_0$ except in the case when $r = 2$ and $C_0$ is a single edge.

We now recolor $G'$ with colors from $K \cup \{3\}$. Let $G$ have bipartition $V(G) = A \cup B$. We shall provisionally color the vertices of $A$ with $a \in K$ and the vertices of $B$ with $b \in K$. We shall color the edges of $G$ in such a way that (with a few exceptions) each $v \in A$ with $d_{G'}(v) \geq 2$ will be colored so that $S(v) \cap K \in S_A$, while

$$S_A = \{ \{0, a\}, \{a, b\}, \{b, c\}, \{0, a, c\}, \{0, b, c\} \},$$

while for $v \in B$, $S(v) \cap K \in S_B$, where

$$S_B = \{ \{0, b\}, \{0, c\}, \{a, c\}, \{0, a, b\}, \{a, b, c\} \}.$$

This is sufficient, since if $uv \in E(G)$, $u \in A$, $v \in B$, and $S(u) = S(v)$, then $S(u) \cap K = S(v) \cap K$. But $S_A \cap S_B = \emptyset$, so $d_{G'}(u) < 2$, say. But then $|S(v) \cap K| = |S(u) \cap K| < 2$, so $S(v) \cap K \notin S_A, S_B$ and $d_{G'}(v) < 2$. However, $E(G') \supseteq E(G\{1,2\})$, so if $uv \notin E(G\{1,2\})$, then $S(u) \cap \{4, \ldots, \chi\} \neq S(v) \cap \{4, \ldots, \chi\}$ by condition D and the fact that we do not recolor any edges of $G\{4, \ldots, \chi\}$.

We shall now color each component of $G'$ independently.

**Case 1. Good isolated paths.** Using the elements of $K$, color the edges of a good path arbitrarily so that the sum of the two colors meeting a degree 2 vertex of the path is equal to the color of this vertex. Any degree 2 vertex $v$ will have $S(v) \cap K \in \{\{0, a\}, \{b, c\}\} \subseteq S_A$ if $v \in A$ and $S(v) \cap K \in \{\{0, b\}, \{a, c\}\} \subseteq S_B$ if $v \in B$. 


Case 2. Good isolated cycles. If the cycle length is divisible by 4, then we can color the edges from \( k \) so that the sum of the two colors meeting a vertex \( v \) is equal to the vertex color in \( K \). Any color 3 chord will remain colored 3. If the cycle length is not divisible by 4 and there are no color 3 chords, then none of the vertices meet color 3 in \( G \). However, by condition B of Lemma 3.1 all the neighbors of vertices of the cycle meet all three colors \( \{1, 2, 3\} \) in \( G \). If we give the cycle any avd-coloring using colors from \( K \), we are done since every vertex on the cycle will meet only two colors from \( K \cup \{3\} \), whereas their neighbors off the cycle will meet three such colors. (This is one case where we do not insist that \( S(v) \cap k \) lies in \( S_A \) or \( S_B \).) Finally, if the cycle has a color 3 chord \( uv \), recolor \( u \) and a neighbor \( w \) of \( u \) on \( C \) with color \( c \). Now color the edges around the cycle so that \( u \) meets \( \{0, c\} \) if \( u \in A \) or \( \{a, b\} \) if \( u \in B \). Then \( v \) is still labeled with \( a \) or \( b \) so the chord \( uv \) can be recolored by some color of \( K \), making the coloring on \( C \) proper (see Figure 7). It is easily checked that \( S(v) \cap K \), \( S(u) \cap K \), and \( S(w) \cap K \) lie in the correct set \( S_A \) or \( S_B \) as required.

Case 3. Stars of components. Remove any edges from \( G' \) that are chords of some component cycle \( C_i \) of the star. These edges will remain colored 3. If the central component \( C_0 \) is a cycle of length 2 mod 4, relabel one (and only one) vertex \( u \in C_0 \) that is adjacent to an end-component with \( b \) if \( u \in A \) and \( a \) if \( u \in B \). Assume now that \( C_0 \) is not a single edge. Color the central component so that the sum of two colors meeting a degree 2 vertex of \( C_0 \) is the vertex color of this vertex. Now recolor the color 3 edges \( uv \) from \( C_0 \) to \( C_i \) (\( u \in C_0 , v \in C_i \)) with either 0 or \( c \) if \( u \in A \), or \( a \) or \( b \) if \( u \in B \). Now for each degree 2 or 3 vertex \( u \) of \( C_0 \), \( S(u) \cap K \in S_A \) if \( u \in A \) and \( S(u) \cap K \in S_B \) if \( u \in B \) (see Figure 8).

Each end-component is either a bad cycle of length 2 mod 4, or a single edge. For each end-component that is a cycle \( C \), let \( v \) be the vertex of \( C \) joined to the central component \( C_0 \). Recolor \( v \) and a neighbor of \( v \) on \( C \) with color \( c \in K \). Now color the edges of \( C \) so that the colors of \( K \) meeting \( v \) on \( C \) are \( \{0, c\} \) if \( v \in A \) and \( \{a, b\} \) if \( v \in B \). Now for each degree 2 or 3 vertex \( w \) of \( C \), \( S(w) \cap K \in S_A \) if \( w \in A \) and \( S(w) \cap K \in S_B \) if \( w \in S_B \).
For each end-component $vw$ that is a single edge, let $uv$ be the edge joining $vw$ to $C_0$ (see Figure 8). If $v \in A$, then $uv$ has been colored $a$ or $b$. For either choice there is a choice of 0 or $c$ on edge $vw$ for which $S(v) \cap K \in S_A$. Similarly, if $v \in B$, then $uv$ has been colored 0 or $c$. For either choice there is a choice of $a$ or $b$ on edge $vw$ for which $S(v) \cap K \in S_B$.

Finally, assume $C_0$ is a single edge $xy$. Then $C_0$ is joined to two components, which by condition A must be cycles (see Figure 9). Recolor the edge $xy$ with color 3. (Note that both $x$ and $y$ meet color 3 in the original coloring.) Now if we color the edges to the end-components and the edges of the end-components as before, we obtain a coloring with $x$ distinguished from $y$. In this case $S(x) \cap K$ and $S(y) \cap K$ are not in $S_A$ or $S_B$, so we need to check that $x$ and $y$ are distinguished from all neighbors in $G$. Clearly $x$ and $y$ are distinguished from their neighbors in $G'$. If, say, $zx \in E(G)$, then by condition A, $\{1,2\} \subseteq S(z)$ in the original coloring. Hence in the final coloring $|S(z) \cap K| \geq 2 > |S(x) \cap K|$, so $S(z) \neq S(x)$.

Note that in the proof of Lemma 3.1 we only recolored the edges colored 1, 2, or 3, and for each edge $uv$, either the vertices $u$ and $v$ are distinguished by the colors in $K \cup \{3\}$, or $uv$ is one of the isolated edges of $G\{1,2,3\}$ in condition D of Lemma 3.1.

4. General graphs. The bound in Theorem 1.3 will be obtained by decomposing a general graph into bipartite graphs (Lemma 4.1), and by using an extended version of Lemma 3.1 that makes it possible to color these bipartite graphs “simultaneously” (Lemma 4.2).

**Lemma 4.1.** If $G$ is a $k$-chromatic graph with no isolated edge or isolated $K_3$, then $G$ can be written as the edge disjoint union of $\lceil \log_2 k \rceil$ bipartite graphs, each of which has no isolated edge.

**Proof.** Let $r = \lceil \log_2 k \rceil$. Then $k \leq 2^r$. We first show that $G$ is the union of $r$ bipartite graphs without the restriction on isolated edges. For $r = 1$ this is clear. For $r > 1$ write $V(G)$ as the union of $k$ independent color classes $V_1, \ldots, V_k$. Partition the classes into two groups $V_1, \ldots, V_{\lfloor k/2 \rfloor}$ and $V_{\lfloor k/2 \rfloor + 1}, \ldots, V_k$. Let $G_1$ be the bipartite graph formed by taking all edges from the first set of color classes to the second. Then $G \setminus E(G_1)$ has chromatic number at most $\lceil k/2 \rceil \leq 2^{r-1}$. Hence, by induction, $G \setminus E(G_1)$ can be written as the edge disjoint union of $r-1$ bipartite graphs $G_2, \ldots, G_r$. Thus $G$ is the union of $r$ bipartite graphs as required.

Write $G$ as a union of $r$ bipartite graphs in such a way that the total number of isolated edges in the subgraphs $G_i$ is minimized. Suppose there is an isolated edge $xy$ in $G_1$, say. Since there are no isolated edges in $G$, there must be some other bipartite graph $G_2$, with some edge incident to $x$, say. If we can add $xy$ to $G_2$ without creating an odd cycle, then by removing $xy$ from $G_1$ and adding it to $G_2$ we reduce the number of isolated edges. Hence we may assume there is an even length path from $x$ to $y$ in $G_2$.  

![Fig. 9. Central component is a single edge.](image-url)
If there are edges $xz$ of $G_2$ with $d_{G_2}(z) = 1$, then remove one such edge from $G_2$ and add it to $G_1$. Since there is an even length path from $x$ to $y$, no isolated edges are formed in $G_2$, but there are fewer isolated edges now in $G_1$. Similarly we are done if there are edges $yz$ of $G_2$ with $d_{G_2}(z) = 1$. If no such edge $xz$ or $yz$ exists, remove an edge of an even length path from $x$ to $y$ in $G_2$ and add it to $G_1$. Use the edge of this path incident to $y$ if $d_{G_2}(x) > 1$; otherwise use the edge incident to $x$. This will reduce the total number of isolated edges, except in the case when $G_2$ contains a component consisting of a path $xzy$ of length 2 from $x$ to $y$.

Since $G$ does not contain an isolated $K_3$ there must be some other edge meeting $\{x, y, z\}$ in $G$. Suppose such an edge is incident to either $x$ or $y$. Then this edge must lie in some other bipartite subgraph, say $G_3$. Considering $G_3$ in place of $G_2$ we may assume $G_3$ has a component $xwy$ which is a path of length 2 from $x$ to $y$. In this case put edge $wx$ in $G_1$ and $wy$ in $G_2$. Both $G_1$ and $G_2$ remain bipartite and $G_3$ loses a component. The number of isolated edges in $G_1$ decreases, contradicting our choice of decomposition into bipartite graphs.

Hence we may assume $G$ has some other edge meeting $z$, but $d_G(x) = d_G(y) = 2$. The edge meeting $z$ lies in $G_i$ where $i = 1$ or $i > 2$. In this case we can move $xz$ to $G_i$ and $xy$ to $G_2$. Both $G_i$ and $G_2$ remain bipartite and $G_1$ loses the isolated edge $xy$. This reduces the number of isolated edges and contradicts the assumption that there is an isolated edge in some $G_j$. Hence there is a decomposition into $r$ bipartite graphs, each of which has no isolated edge.  

Lemma 4.2. Assume $G$ is a graph which is the edge disjoint union of bipartite graphs $G_1, \ldots, G_r$, each of which has no isolated edge. Then there exists a proper edge-coloring with colors $\{1, \ldots, r, 2, \ldots, 3, 3, \ldots, 3, 4, \ldots, \chi\}$ such that colors 1, 2, and 3 occur only on the edges of $G_i$ and

A. if $uv \in E(G_i) \setminus E(G_i(1, 2))$, then either $\{1, 2\} \subseteq S(u)$ or $\{1, 2\} \subseteq S(v)$;

B. if $C$ is a cycle in $G(1, 2)$ which does not meet color 3 in $G$, then $\{1, 2, \chi\} \subseteq S(v)$ for every neighbor $v$ in $G_i \setminus C$ of any vertex of $C$;

C. if $C$ is a cycle in $G(1, 2)$ which does meet color 3 in $G$, then there exists $\{1, 2\} \subseteq S(v)$ and $uv \in E(G(3))$ with $\{1, 2\} \subseteq S(v)$; and

D. if $uv$ is an isolated edge in $G(1, 2, 3)$, then either $S(u) \cap \{4, \ldots, \chi\} \neq \emptyset$ or there is an edge in $G$ incident to $u$ colored with color 4.

Proof. By coloring $G$ with $\{1, \ldots, \chi\}$ and splitting colors 1, 2, and 3 into 1, 2, and 3, according to which $G_i$ the edge belongs to, we can find a coloring with the given set of colors so that edges colored 1, 2, or 3, occur only in $G_i$. For all such colorings pick one such that

1. $G\{1, 2, 1, 2, \ldots, 1\}$ has maximal edge count;

2. subject to (1), the sum over $i$ of the number of components of $G(1, 2)$ is minimal;

3. subject to (1)–(2), $G\{3, \ldots, 3\}$ has maximal edge count; and

4. subject to (1)–(3), the number of edges $uv$ failing condition D (for any $i$) is minimal.

As in the proof of Lemma 3.1, we see that conditions A–C hold for each $i$. It remains to prove condition D. Let $u_1v_1$ be an isolated edge of $G\{1, 2, 3\}$. Since $G_i$ contains no isolated edge, we can assume that $d_{G_i}(u_1) \geq 2$ and that $u_1$ meets another color $k > 4$ on some edge of $G_i$. Swap colors 4 and $k$ along a Kempe chain (in $G$) starting at $u_1$. This will reduce the number of edges failing condition D unless the other end-vertex $v_2$ of this chain lies in some isolated edge $u_2v_2$ of $G\{1, 2, 3\}$ and after the recoloring $u_2v_2$ fails condition D. In this case $u_2$ also meets color $k$, so we can form a
new Kempe chain starting at $u_2$ using colors 4 and $k$. Repeating this process we get a sequence of Kempe chains on colors 4 and $k$ from $u_i$ to $v_{i+1}$. Eventually this process must terminate with a coloring reducing the number of edges failing condition D, or with some $v_r = v_1$. However, in this last case recoloring all these Kempe chains makes both $v_1$ and $u_1$ meet color 4.

Proof of Theorem 1.3. Since $K_3$ has a 3-avd-coloring, we can assume $G$ contains no $K_3$ component. Decompose $G$ using Lemma 4.1 and color $G$ as in Lemma 4.2. Now recolor each bipartite subgraph $G_i$, replacing $1_i$, $2_i$, $3_i$ with a set of five colors $K_i = \{0, a_i, b_i, c_i, d_i\}$, disjoint for each $i$, as in the proof of Theorem 1.2. Some edges $uv$ of $G_i$ may be isolated in $G_i \{1_i, 2_i, 3_i\}$, so $u$ and $v$ will not necessarily be distinguished in $G_i$; however, for all other edges $uv \in E(G_i)$, $S(u) \cap K_i \neq S(v) \cap K_i$ by the comment at the end of section 3. Also, this recoloring does not change any of the colors 4, ..., $\chi'$ on $G$. Hence by condition D, if $uv \in E(G_i)$ and $S(u) \cap K_i = S(v) \cap K_i$, then either $S(u) \cap \{4, \ldots, \chi'\} \neq S(v) \cap \{4, \ldots, \chi'\}$ or $4 \in S(u)$. Let $H$ be the subgraph of edges $uv \in E(G)$ such that $u$ and $v$ are not distinguished by the colors in $K_i \cup \{4, \ldots, \chi'\}$, where $uv \in E(G_i)$. Let $H_1$ be the subgraph of $H$ consisting of all the isolated edges of $H$. Each nonisolated vertex in $H$ meets color 4, so $G\{4\} \cup H_1$ forms a collection of paths and cycles with all edges of $H_1$ on the interior of any path or cycle. Split color 4 into three colors $4_A$, $4_B$, and $4_C$. By alternately changing 4 into $4_A$ or $4_B$ along the paths and cycles of $G\{4\} \cup H_1$ we can distinguish the end-vertices of each edge of $H_1$. If a cycle of length 2 mod 4 occurs, we shall also need to color some of the color 4 edges of this cycle with $4_C$. All other color 4 edges in $G$ may become $4_C$ without loss of generality. This increases the number of colors used by 2 and distinguishes $u$ and $v$ for all $uv \in E(H_1)$. The graph $H_C = H \setminus H_1$ has no isolated edge, and $\Delta(H_C) \leq r \leq \lfloor \log_2 k \rfloor$. Pick $\chi'_a(H_C)$ new colors and recolor $H_C$ so that it has an avd-coloring using these colors. The resulting coloring is avd. To see this, pick any edge $uv$ of $G$. If $uv \in E(G_i)$ and $uv \notin E(H)$, then $S(u) \cap K_i \neq S(v) \cap K_i$ or $S(u) \cap \{4_A, 4_B, 4_C, 5, \ldots, \chi'\} \neq S(v) \cap \{4_A, 4_B, 4_C, 5, \ldots, \chi'\}$ since the recoloring of $H_C$ removes elements from $S(u) \cap K_i$ only when $u$ is in an isolated edge of $G_i \{1_i, 2_i, 3_i\}$. But in this case $|S(v) \cap K_i| \geq 2$ (by condition A) and $|S(u) \cap K_i| = 0$. If $uv \in E(H_1)$, then $S(u) \cap \{4_A, 4_B, 4_C\} \neq S(v) \cap \{4_A, 4_B, 4_C\}$, and if $uv \in E(H_C)$, then $u$ and $v$ are distinguished by the $\chi'_a(H_C)$ new colors.

Thus $\chi'_a(G) \leq \chi'(G) - 3 + 5r + 2 + \chi'_a(H_C)$. Finally, $\Delta(H_C) \leq \chi'(H_C) \leq r < \Delta(G)$. So by induction on $\Delta(G)$ we may assume $\chi'_a(H_C) = r + O(\log r)$, and $\chi'_a(G) = \Delta(G) + O(r) = \Delta(G) + O(\log k)$.

REFERENCES