Decompositions of graphs into cycles with chords

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In memory of Dick Schelp, who passed away shortly after the submission of this paper.

Abstract

We show that if $G$ is a graph on at least $3r + 4s$ vertices with minimum degree at least $2r + 3s$, then $G$ contains $r + s$ vertex disjoint cycles, where each of $s$ of these cycles either contain two chords, or are of order 4 and contain one chord.

Keywords: graphs, cycles, decompositions.

1 Introduction and Main Result

The following beautiful conjecture of Bialostochi, Finkel, and Gyárfás appears in [1].
Conjecture 1. Let \( r, s \) be nonnegative integers and let \( G \) be a graph with \( |V(G)| \geq 3r + 4s \) and minimum degree \( \delta(G) \geq 2r + 3s \). Then \( G \) contains a collection of \( r \) cycles and \( s \) chorded cycles, all vertex disjoint.

The complete bipartite graph \( K_{2r+3s-1,n-2r-3s+1} \) shows that the minimum degree cannot be lowered when \( n \geq 4r + 6s - 2 \).

The conjecture is a generalization of well known results of Pósa, and of Corrádi and Hajnal. Pósa proved (see [7, problem 10.2]) that any graph with minimum degree at least 3 contains a chorded cycle and Corrádi and Hajnal [3] proved that any graph of minimum degree at least \( 2r \) of order \( n \geq 3r \) contains \( r \) vertex disjoint cycles.

The purpose of this article is to show that a stronger result than that given in the conjecture is true. We prove the following theorem.

**Theorem 2.** If \( G \) is a simple graph on \( |V(G)| \geq 3r + 4s \) vertices with \( \delta(G) \geq 2r + 3s \), then \( G \) contains \( r + s \) vertex disjoint cycles, each of \( s \) of them either with two chords, or a \( C_4 \) with one chord.

It is likely the case that among the chorded “long” cycles more than two chords will be present, and that one can insist on two chords even in the \( C_4 \)s, but our method of proof does not establish this.

It has come to our attention that Conjecture 1 has been proved by Gao, Li, and Yan and appears in [4], but they do not address the stronger result given in our theorem. Also a degree sum condition is used by Chiba, Fujita, Gao, and Li in [2], and neighborhood union conditions are used by Gao, Li, and Yan [5], and by Qiao [8], to realize disjoint chorded cycles. Finally, independently of our results, Gould, Hirohata, and Horn [6] proved a result on the existence of disjoint doubly chorded cycles under a degree sum condition. This result however only applies to the \( r = 0 \) case with \( |V(G)| \geq 6s \).

The proof of our theorem is based on several technical theorems and lemmas, the last two of which are in themselves of special interest. One of these (Theorem 12) generalizes the result of Pósa mentioned earlier by showing that a graph with minimum degree 3 contains a cycle with two chords.

We write as usual \( P_n, C_n, K_n, \) or \( E_n \) for a path, cycle, complete graph, or empty graph respectively on \( n \) vertices. When the number of vertices is
unspecified we shall write for example $P$ or $C$. We denote by $C_{n+3}$ any cycle of length $n$ with at least $k$ additional chords, and $K_n^{-k}$ the complete graph on $n$ vertices with at most $k$ edges removed. If $k = 1$ then we write just $C^+_n$ or $K_n^-$ for brevity. Note that, for example, a $C_{n+3}$ graph is also considered as a special case of a $C_{n+2}$ graph. It will also be convenient to denote by $C^+_n$ a graph that is either a $C^+_{n+2}$ or a $C^+_4$. We shall write $G \cup H$ for the vertex disjoint union of $G$ and $H$, and $G \cup H + ke$ for such a graph with $k$ additional edges added between $G$ and $H$. We shall also use the notation $H \subseteq G$ or $G \supseteq H$ to indicate that $H$ is a subgraph of $G$, and $H \subset G$ or $G \supset H$ to indicate that $H$ is a non-spanning subgraph of $G$, i.e., a subgraph with $|V(H)| < |V(G)|$. For example, the statement $G \supset C^+_k$ indicates that $G$ contains a non-hamiltonian cycle. We shall occasionally abuse notation by regarding a subgraph $H \subseteq G$ also as a subset of vertices of $G$. So for example, we write $G[\bar{H} \cup v]$ instead of $G[V(H) \cup \{v\}]$ for the subgraph induced by the vertices of $H$ and an extra vertex $v \in V(G)$.

The proof of Theorem 2 involves a number of technical theorems and lemmas, the relevance of which only becomes apparent in the proof of Theorem 2. Thus we shall first give the proof of Theorem 2 assuming these results, and then state and prove them in the next section.

Proof of Theorem 2. Consider all possible decompositions of the graph $G$ into $r+s$ vertex disjoint subgraphs $G_i$, $i = 1, \ldots, r+s$, each subgraph being of one of the following types

$$C^+_n, C^{+2}_n, C^+_4, K_4, K_5, K_6^-, K_7^{-3},$$

and possibly an additional set $S$ of unused vertices. By [3] there is a collection of $r+s$ disjoint cycles in $G$, so at least one such decomposition exists. Among these decompositions, pick one with the minimal number of $C$-s. Say there are $r'$ $C$-s and $s' = r+s-r'$ of the other subgraphs. If $r' \leq r$ then we are done, as each of the other graphs on this list contains (and hence can be replaced with) a cycle while $K_4$, $K_5$, $K_6^-$ and $K_7^{-3}$ all contain a $C^+_4$ subgraph. Hence we may assume $r' > r$. Among the decompositions with this minimal $r'$, we shall take one of minimal weight, where the weight of a decomposition is defined as the sum of certain weights assigned to each of the subgraphs $G_i$. The weights $w(G_i)$ assigned to these subgraphs are given in Table 1. Here $\varepsilon$ is chosen so that $0 < \varepsilon < \frac{1}{7}$ and we regard a subgraph as a $C$ only if it fails to have enough chords to be a $C^+_4$. We call such a decomposition with
Table 1: Weights of graphs ($0 < \varepsilon < \frac{1}{7}$).

<table>
<thead>
<tr>
<th>$G_i$</th>
<th>$w(G_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_n$, $n \geq 3$</td>
<td>$n$</td>
</tr>
<tr>
<td>$C_n^{+2}$, $n \geq 5$</td>
<td>$n$</td>
</tr>
<tr>
<td>$C_4^+$</td>
<td>4</td>
</tr>
<tr>
<td>$K_4$</td>
<td>$4 - 4\varepsilon$</td>
</tr>
<tr>
<td>$K_5$</td>
<td>$4 - 5\varepsilon$</td>
</tr>
<tr>
<td>$K_6^{-}$</td>
<td>$4 - 6\varepsilon$</td>
</tr>
<tr>
<td>$K_7^{-3}$</td>
<td>$4 - 7\varepsilon$</td>
</tr>
</tbody>
</table>

this $r'$ and minimal weight an optimal decomposition. Note that as we are assuming for contradiction that $r' > r$, there will always be at least one $C_*$ in our optimal decomposition.

**Claim 1:** $S = \emptyset$ in any optimal decomposition.

**Proof of Claim 1.** Suppose otherwise and pick $v \in S$. By Lemma 4 below, $v$ can send at most 3 edges to any $G_i \neq C_*$, otherwise we could construct a new decomposition replacing $G_i$ with a $G_i' \neq C_*$ of smaller weight, or two $C_4^+$s, which on discarding a $C_*$ from our decomposition would result in a decomposition with smaller $r'$. Similarly, by Lemma 3, $v$ can send at most 2 edges to any $G_i = C_*$. Let $G_1$ be one of the $C_*$ cycles. Then $d(v) \leq 2(r'-1) + 3s' + d_{S \cup G_1}(v)$ where $d_{S \cup G_1}(v)$ is the degree of $v$ in the subgraph $G[S \cup G_1]$. But $d(v) \geq 2r + 3s \geq 2r' + 3s' + 1$. Thus $d_{S \cup G_1}(v) \geq 3$ for every $v \in S$. But then by Theorem 13, $G[S \cup G_1]$ contains either a cycle of smaller order than $G_1$, or a $C_*^{+2}$, either of which could be swapped with $G_1$ to obtain a better decomposition. Thus $S = \emptyset$. □

**Claim 2:** There are no $C_n$s with $n > 3$ and no $C_n^{+2}$s with $n > 4$ in any optimal decomposition.

**Proof of Claim 2.** Consider a subgraph $G_i$ with maximal weight in an optimal decomposition of $G$. Suppose $G_i = C_n$ with $n \geq 4$. The sum of the degrees $d_{G_i}(v)$ over $v \in G_i$ is at most $2n + 2$ as otherwise $G_i$ would have two chords. If $G_i$ sends $2n + 1$ edges to any $G_j = C_*$, $j \neq i$, then by Lemma 10 we can replace $G_i$ and $G_j$ in the decomposition with two disjoint $C_*$s with
smaller total weight (when \( n = 4 \)) or a \( C_4 \) and a \( C_4^\dagger \) (when \( n \geq 5 \)) giving a decomposition with smaller \( r' \). Note that we are assuming \( G_i \) has maximal weight so that \( |V(G_j)| \leq |V(G_i)| \). Again by Lemma 10, if \( G_i \) sends \( 3n + 1 \) (\( \geq 2n + 1 \) for \( n \geq 5 \) or \( \geq 11 \) for \( n = 4 \)) edges to any \( G_j = C_4^\dagger \) (including \( K_4 = C_4^{+2} \) as a special case), then we can replace \( G_i \cup G_j \) with a \( C_4 \cup C_4^\dagger \) with fewer total number of vertices and hence smaller total weight. If \( G_i \) sends \( 3n+1 \geq 13 \) edges to a \( G_j \in \{K_5, K_6^-\} \), then there must be some vertex \( v \in G_j \) that sends at least 3 edges to \( G_i \). Then by Lemma 3, \( G[G_i \cup v] \) contains a shorter cycle \( C \) than \( G_i \), while \( G_j - v \) contains a \( K_4 \), which has weight at most \( 2\varepsilon \) more than \( G_j \). Replacing \( G_i \) and \( G_j \) in the decomposition by \( C \) and this \( K_4 \) gives a decomposition with at least \( 1 - 2\varepsilon \) smaller weight. Now suppose \( G_i \) sends \( 3n + 1 \) edges to \( G_j = K_7^{-3} \). Then at least one vertex \( v \in G_i \) sends at least 4 edges to \( K_7^{-3} \), so by Lemma 4 we can decompose \( G[v \cup G_j] \) into two \( C_4^- \)'s. Replacing \( G_i \) and \( G_j \) by these gives a decomposition with smaller \( r' \). Since \( S = \emptyset \), and combining the above bounds, the sum of the degrees \( d(v) \) in \( G \) of the vertices of \( G_i \) is at most \((2n + 2) + 2n(r' - 1) + 3ns' < (2r + 3s)n \), so at least one vertex of \( G_i \) violates the minimum degree condition.

Now suppose the subgraph with maximal weight is \( G_i = C_4^{+2} \) for some \( n > 4 \). As \( C_5^{+2} \supset C_4^{+} \) and \( w(C_5^{+2}) > w(C_4^{+}) \) we may assume \( n \geq 6 \). The sum of the degrees \( d_{G_i}(v) \) over \( v \in G_i \) is at most \( 3n \) as otherwise \( G_i \) would have more than \( n/2 \) chords and by Lemma 5 we could replace \( G_i \) by a subgraph of \( G_i \) of smaller weight. If \( G_i \) sends \( 2n + 1 \) edges to any \( G_j = C_4 \), then by Lemma 10 we can replace \( G_i \cup G_j \) in the decomposition with a \( C_4 \cup C_4^\dagger \) with fewer total number of vertices and hence smaller weight. Note that \( |V(G_j)| \leq |V(G_i)| \) as \( G_i \) has maximal weight. Similarly, if \( G_i \) sends \( 3n + 1 \) edges to any \( G_j = C_4^{+} \) (including \( K_4 = C_4^{+2} \)) then, by Lemma 11, we can replace \( G_i \cup G_j \) with a \( C_4^\dagger \cup C_4^\dagger \) with fewer total number of vertices and hence smaller weight. If \( G_i \) sends at least \( 3n + 1 \geq 19 \) edges to a \( G_j \in \{K_5, K_6^-\} \), then there must be some vertex \( v \in G_j \) that sends at least 4 edges to \( G_i \). Then by Lemma 4, \( G[G_i \cup v] \) contains \( C_4^\dagger \) on fewer vertices than \( G_i \), while \( G_j - v \) contains a \( K_4 \), which has weight at most \( 2\varepsilon \) more than \( G_j \). Replacing \( G_i \) and \( G_j \) in the decomposition by this \( C_4^\dagger \) and \( K_4 \) gives a decomposition with at least \( 1 - 2\varepsilon \) smaller weight. Now suppose \( G_i \) sends at least \( 3n + 1 \) edges to \( G_j = K_7^{-3} \). Then some vertex \( v \in G_i \) sends at least 4 edges to \( K_7^{-3} \), so by Lemma 4 we can decompose \( G[v \cup G_j] \) into two \( C_4^- \)'s. Replacing \( G_i \cup G_j \) by these gives a decomposition with smaller weight (8 versus \( n + 4 - 7\varepsilon \geq 10 - 7\varepsilon > 9 \)). Combining the above bounds and using \( S = \emptyset \), the sum of the degrees in \( G \) of the vertices of \( G_i \) is at most \( 2nr' + 3ns' < (2r + 3s)n \), so at least one vertex of \( G_i \) violates
the minimum degree condition.

By Claims 1 and 2, the optimal decomposition consists only of graphs in the set \( \{ C_3, C_4^+, K_4, K_5, K_6^-, K_7^{-3} \} \) and the \( G_i \)s cover all vertices of \( G \). As \(|V(G)| \geq 3r + 4s\) and \( r' > r \), we must have at least one subgraph \( G_i \in \{ K_5, K_6^-, K_7^{-3} \} \). We now construct an “almost optimal” decomposition with a set \( S \) of unused vertices with \( 1 \leq |S| \leq 3 \) and with no \(|V(G_i)| > 4\) unless \(|S| = 3\). If the optimal decomposition contains \( K_7^{-3} \), then remove three vertices from it to produce a \( K_4 \). The \( S \) will then consist of the three removed vertices. If there is no \( K_7^{-3} \) then remove one vertex from a \( K_6^- \) to produce a \( K_5 \), or one from a \( K_5 \) to produce a \( K_4 \). Repeating this process at most three times and placing the removed vertices in \( S \), we obtain a decomposition with a set \( S \) of unused vertices with either \(|S| = 3\), or with \(|S| < 3\) and no \(|V(G_i)| > 4\). Note that although this decomposition no longer has minimal weight, its weight is in all cases at most \( 3\varepsilon \) larger than the optimal decomposition, and it has the same value of \( r' \). As \( r' > 0 \) we also have, say, \( G_1 = C_3 \). We count the sum of the degrees of the vertices in \( S \cup G_1 \) with the vertices in \( S \) weighted by a factor of \( \frac{3}{|S|} \), i.e., we estimate

\[
E = \frac{3}{|S|} \sum_{v \in S} d(v) + \sum_{v \in V(G_1)} d(v).
\]

Suppose \( G_j = C_3, j > 1 \). Each \( v \in S \) can send only one edge to \( G_j \), otherwise \( G[v \cup G_j] \) would contain a \( C_4^+ \) that we could swap with \( G_j \) giving a decomposition with smaller \( r' \) (and hence a smaller \( r' \) than the optimal decomposition). Clearly \( G_1 = C_3 \) can send at most 9 edges to \( G_j = C_3 \), so we have a contribution of at most \( 3 + 9 = 12 \) to \( E \) from edges from \( S \cup G_1 \) to this \( G_j \).

Now suppose \( G_j \in \{ C_4^+, K_4 \} \) and suppose further that there is a contribution to \( E \) of more than 18 from edges between \( S \cup G_1 \) and \( G_j \). Clearly there are at most 12 edges between \( G_1 = C_3 \) and \( G_j \), so there must be a contribution of more than 6 to \( E \) from the edges between \( S \) and \( G_j \). Thus there exists \( v \in S \) which sends more than 2 edges to \( G_j \). If \( G_j = C_4^+ \) and \( G[v \cup G_j] \supseteq K_4 \) then we can swap \( G_j \) with this \( K_4 \), decreasing the weight by \( 4\varepsilon \). This would give a better decomposition than the optimal one, a contradiction. Hence we may assume \( v \) is joined to both degree 2 vertices and one of the degree 3 vertices in \( G_j \) (see Figure 1). Any single vertex \( u \) of \( G_j \) can be removed
from $G[v \cup G_j]$ yielding a $C_4^+$ except possibly for $v_0$, a degree 3 vertex of $G_j$ that is joined to $v$. If $G_j = K_4$ then any vertex of $G[v \cup G_j]$ can be removed yielding a $C_4^+$ or $K_4$. Thus if there are more than $3 + 1 + 1 + 1 = 6$ edges from $G_j$ to $G_1$ we can find a $u \in G_j - v_0$ joined to two vertices of $G_1$ forming a $C_4^+$ on $G_1 \cup u$ and a $C_4^+$ on $v \cup G_j - u$. Replacing $G_1 \cup G_j$ with these two $C_4^+$s gives a decomposition with smaller $r'$, so we may assume that there are at most 6 edges from $G_1$ to $G_j$. But there at most 4 edges from each $v \in S$ to $G_j$. Thus we have a contribution of at most $3 \times 4 + 6 = 18$ to $E$ from the edges to $G_j$.

Now suppose $G_j \in \{K_5, K_6^-, K_7^-\}$. Then $|S| = 3$. By Lemma 6 there can be at most $12 + 5 = 17$ edges from $S \cup G_1$ to a $K_5$, otherwise we could find two $C_4^+$s in $G[S \cup G_1 \cup G_j]$ which could be swapped with $G_1 \cup G_j$ giving a decomposition with smaller $r'$. Similarly there are at most $12 + 6 = 18$ edges from $S \cup G_1$ to a $K_6^-$ and at most $9 + 7 = 16$ edges from $S \cup G_1$ to a $K_7^-$. Hence there is a contribution of at most 18 to $E$ from edges to this $G_j$.

Each $v \in S$ can send at most one edge to $G_1$, otherwise we would have a $C_4^+$ which could be swapped with $G_1$, and each $v \in S$ can send at most $|S| - 1 \leq 2$ edges to $S$. Thus the contribution to $E$ from edges within $G[S \cup G_1]$ is at most 6 from edges in $G_1$, plus 6 from edges between $G_1$ and $S$, and 6 from edges in $S$. Equality occurs only if $|S| = 3$ and $G[S \cup G_1]$ is two triangles with three edges joining them. But in this case either $G[S \cup G_1] \supseteq C_6^{2}$ (if the three edges form a matching) or $G[S \cup G_1] \supseteq C_4^+$ (otherwise). This $C_6^{2}$ or $C_4^+$ can be swapped with $G_1$ reducing $r'$, a contradiction. Hence the contribution to $E$ from edges in $G[S \cup G_1]$ is at most 17.

In total, we have $E \leq 12(r' - 1) + 18s' + 17 < 6(2r + 3s)$ contradicting the fact that $E \geq 6\delta(G) \geq 6(2r + 3s)$. Thus a decomposition exists with $r' = r$ and the theorem is proved. \qed
2 Technical lemmas

Lemma 3. If \( G = E_1 \cup C_n + 3e \) then either \( G \supseteq C_m \) for some \( m < n \) or \( n = 3 \) and \( G = K_4 \).

Proof. Let \( E_1 = \{v\} \). If \( v \) sends three edges to \( C_n \) then the shortest arc between neighbors of \( v \) on \( C_n \) is of edge length at most \( n/3 \) and hence there is a cycle through \( v \) of length at most \( n/3 + 2 \). If \( n > 3 \) then we are done as \( n/3 + 2 < n \). If \( n = 3 \) then \( G = K_4 \). \( \square \)

Lemma 4. For \( n > 4 \), \( E_1 \cup C_n^+ + 4e \supseteq C_m^+ \) for some \( m < n \). Also \( E_1 \cup C_4^+ + 4e \supseteq K_4 \), \( E_1 \cup K_4 + 4e = K_5 \), \( E_1 \cup K_5 + 4e = K_6^+ \), \( E_1 \cup K_6^- + 4e = K_7^{-3} \), and \( E_1 \cup K_7^{-3} + 4e \supseteq C_4^+ \cup C_4^+ \).

Proof. Suppose first that \( H = C_n^+2 \) with \( n > 4 \) and write \( E_1 = \{v\} \). Let \( vv_i, i = 1, \ldots, 4 \), be the edges from \( v \) to \( H \) with \( v_i \) ordered cyclically around the main cycle \( C \) of \( H \). If the arc between any consecutive \( v_i \) on \( C \), say \( v_4 \) and \( v_1 \), contains at least two interior vertices, then \( vv_1 \ldots vv_2 \ldots vv_3 \ldots vv_4 \) is a shorter cycle with two chords \( vv_2, vv_3 \) (Figure 2(a)). Thus we may assume there is at most one vertex of \( C \) between each consecutive pair of \( v_i \), say \( C = v_1v_2v_3v_4v_1 \) where some of the \( v_i \) may not exist. If three consecutive \( v_i \) exist with no \( u_i \) between them then they form a \( C_4^+ \) with \( v \) (Figure 2(b)). Hence we may assume that no two consecutive \( u_i \) are both missing.

Now consider the chords of \( H \). If one of these chords joins a pair of consecutive \( v_i \)s, say \( v_1v_2 \) then \( \{v, v_1, v_2, u_1\} \) induces a \( C_4^+ \) (Figure 2(c)). If one of the chords joins opposite \( v_i \), say \( v_1v_3 \), then \( vv_1v_2v_2v_3v \) is cycle with two chords \( v_1v_3 \) and \( vv_2 \) (Figure 2(d)). Note that either \( u_3 \) or \( u_4 \) exists, so this cycle has length less than \( n \). If a chord joins a \( u_i \) to a \( v_j \), say \( u_1 \) to \( v_3 \) then \( vv_1u_1v_2u_2v_3v \) is a cycle with chords \( u_1v_2 \) and \( vv_3 \) (Figure 2(e)). Once again, either \( u_3 \) or \( u_4 \) exists so this cycle has length less than \( n \). Thus we may assume both chords are between \( u_i \) vertices. If two consecutive \( u_i \) are joined by a chord, say \( u_1u_2 \), then \( vv_1u_1v_2u_2v_3v \) is a shorter cycle with chords \( vv_2 \) and \( u_1u_2 \) (Figure 2(f)). The only remaining case is when both \( u_1u_3 \) and \( u_2u_4 \) are chords. But then \( vv_1u_1v_3u_2v_2v \) is a shorter cycle with two chords \( u_1v_2 \) and \( vv_3 \) (Figure 2(g)).
Lemma 5. For a cycle $C$, suppose $C$ contains a $K_4$ (see Figure 1). Also it is clear that $E_1 \cup K_4 + 4e = K_5$, $E_1 \cup K_5 + 4e = K_6^-$, and $E_1 \cup K_6^- + 4e = K_7^{-3}$. Finally, suppose $H = K_7^{-3}$. As $H$ is missing at most three edges and $v$ is joined to 4 vertices of $H$, there must be a triangle $uv_1v_2$ in $G = E_1 \cup H + 4e$. Let $H' = H - \{v_1, v_2\}$. Then $H' = K_5^{-3}$. If $H'$ is missing exactly 3 edges, one can find a vertex $u \in H'$ such that $H' - u \supseteq C_4^+$ (pick $u$ to be a vertex of degree at least two in the complement of $H'$). Similarly, if $H'$ is missing exactly 2 edges then there are at least three choices for $u$ so that $H' - u \supseteq C_4^+$ (any vertex incident to a missing edge). For $H' = K_5^-$, $H' - u$ always contains a $C_4^+$. Thus in general, if $x$ edges are missing from $H'$ there are at most $x + 1$ vertices $u$ such that $H' - u \supseteq C_4^+$. There are also at most $3 - x$ missing edges between $H'$ and $\{v_1, v_2\}$ in $H$, so at most $3 - x$ values of $u$ such that $uv_1v_2v$ is not a $C_4^+$. As $(3 - x) + (1 + x) < 5$, there is a $u$ such that both $uv_1v_2v$ is $C_4^+$ and $H' - u \supseteq C_4^+$ so $G \supseteq C_4^+ \cup C_4^+$. \hfill \square

Figure 2: Finding $H'$ in Lemma 4. Stars indicate possible presence of vertices.

If $H = C_4^+$ then $E_1 \cup H + 4e$ contains a $K_4$ (see Figure 1). Also it is clear that $E_1 \cup K_4 + 4e = K_5$, $E_1 \cup K_5 + 4e = K_6^-$, and $E_1 \cup K_6^- + 4e = K_7^{-3}$. Finally, suppose $H = K_7^{-3}$. As $H$ is missing at most three edges and $v$ is joined to 4 vertices of $H$, there must be a triangle $uv_1v_2$ in $G = E_1 \cup H + 4e$. Let $H' = H - \{v_1, v_2\}$. Then $H' = K_5^{-3}$. If $H'$ is missing exactly 3 edges, one can find a vertex $u \in H'$ such that $H' - u \supseteq C_4^+$ (pick $u$ to be a vertex of degree at least two in the complement of $H'$). Similarly, if $H'$ is missing exactly 2 edges then there are at least three choices for $u$ so that $H' - u \supseteq C_4^+$ (any vertex incident to a missing edge). For $H' = K_5^-$, $H' - u$ always contains a $C_4^+$. Thus in general, if $x$ edges are missing from $H'$ there are at most $x + 1$ vertices $u$ such that $H' - u \supseteq C_4^+$. There are also at most $3 - x$ missing edges between $H'$ and $\{v_1, v_2\}$ in $H$, so at most $3 - x$ values of $u$ such that $uv_1v_2v$ is not a $C_4^+$. As $(3 - x) + (1 + x) < 5$, there is a $u$ such that both $uv_1v_2v$ is $C_4^+$ and $H' - u \supseteq C_4^+$ so $G \supseteq C_4^+ \cup C_4^+$. \hfill \square

Suppose a cycle $C$ has two chords $e_1 = u_1v_1$ and $e_2 = u_2v_2$. We say that $e_1$ and $e_2$ cross if $u_1, u_2, v_1, v_2$ are all distinct and occur in the cyclic order $u_1, u_2, v_1, v_2$ on $C$. Note that incident chords are not considered crossing.

Lemma 5. For $n \geq 8$, $C_n^+ \supset C_s^+$. Also $C_7^+ \supset C_6^+ \supset C_5^+$, $C_n^+ \supset C_s^+$, and $C_{5^2}^+ \supset C_4^+$. \hfill \square

Proof. We prove the first statement, i.e., that $C_n^+ \supset C_m^+$ for some $m < n$. The remaining low order cases can be checked by a case-by-case analysis...
Suppose we are given \( C = C_n \) with five chords. If \( e \) is one chord and two other chords fail to cross \( e \), then we obtain a smaller \( C^*_n + 2 \) by simply shortening the cycle using one of the chords. Let \( e_1 \) and \( e_2 \) be two chords, chosen to be non-crossing if possible. Then the remaining three chords \( e_3, e_4, e_5 \) all cross both \( e_1 \) and \( e_2 \). Suppose \( e_3 \) and \( e_4 \) cross. Then using \( e_3, e_4 \) and two arcs of \( C \) we obtain a cycle with two chords \( e_1, e_2 \). This gives us our \( C^*_n + 2 \) unless this cycle is hamiltonian, i.e., unless \( e_3 \) and \( e_4 \) are adjacent at both ends. A similar argument applies to \( e_3, e_5 \) and \( e_4, e_5 \). But we know at least two of these three pairs are crossing, otherwise one of \( e_3, e_4, e_5 \) would fail to cross two others. But if say \( e_3, e_4 \) and \( e_4, e_5 \) are crossing and adjacent at both ends with \( e_3 \neq e_5 \), then \( e_3, e_5 \) are crossing and not adjacent at both ends. Hence we always obtain a smaller \( C^*_n + 2 \).

**Lemma 6.** Each of the following graphs \( G \) satisfy \( G \supseteq C^*_n + C^*_n + 4 \).

\[
\begin{align*}
C_3 \cup K_5 + 6e & \quad C_3 \cup K_6^- + 7e & \quad C_3 \cup K_7^{-3} + 8e \\
E_3 \cup K_5 + 13e & \quad E_3 \cup K_6^- + 13e & \quad E_3 \cup K_7^{-3} + 10e
\end{align*}
\]

**Proof.** If there are \( |H| + 1 \) edges between a \( C_3 \) and \( H \in \{K_5, K_6^-, K_7^{-3}\} \) then there is a vertex \( v \in H \) joined to two vertices of \( C_3 \) forming a \( C^*_n \). But \( H - v \) always contains a \( C^*_n \).

Now suppose there are 13 edges between an \( E_3 \) and a \( K_5 \) (so only 2 of the possible edges are missing). One of the vertices \( v \in E_3 \) must send 5 edges to the \( K_5 \). Then \( G[(E_3 - v) \cup K_5] = K_7^{-3} \) and \( v \) sends more than 4 edges to this graph, so we are done by Lemma 4.

Suppose there are 13 edges between an \( E_3 \) and a \( K_6^- \). One of the vertices \( v \in E_3 \) must send at least 4 edges to the \( K_6^- \) forming a \( K_7^{-3} \) by Lemma 4. We then have at least \( 13 - 6 = 7 \) edges between the two remaining vertices of \( E_3 \) and this \( K_7^{-3} \). One of these vertices must send at least 4 edges to the \( K_7^{-3} \) forming two \( C^*_n \)'s by Lemma 4.

Finally, suppose there are 10 edges between \( E_3 \) and \( K_7^{-3} \). Then one of the vertices of \( E_3 \) sends at least 4 edges to the \( K_7^{-3} \) forming two \( C^*_n \)'s by Lemma 4.

**Lemma 7.** \( C_n \cup C_n + 4e \supseteq C^*_n + 2 \).

**Proof.** Assume the edges join \( v_i \in C \) to \( v'_i \in C' \), \( i = 1, 2, 3, 4 \), with the \( v_i \) and \( v'_i \) not necessarily distinct. If \( v_1 = v_2 = v_3 = v_4 \) then the \( v'_i \) are distinct.
and there is an arc, say \( P = v'_1 \ldots v'_2 \) of \( C' \) meeting \( v'_3, v'_4 \). Then \( v_1Pv_1 \) is a cycle with chords \( v_3v'_3, v_4v'_4 \). Similarly we are done if all \( v'_i \) are equal. If \( v_1 \neq v_2 = v_3 = v_4 \) then \( v'_2, v'_3, v'_4 \) are distinct, so \( v'_1 \) is distinct from at least two other \( v'_i \). Of course \( v_1 \) is also distinct from at least two other \( v_i \) as well. If no three of the \( v_i \) or \( v'_i \) are equal then it is automatically the case that \( v_1 \) and \( v'_1 \) are distinct from at least two of the other \( v_i \) or \( v'_i \) respectively. Hence we may assume this. Then at least two \( v_i, i > 1 \), are such that there is an arc of \( C \) from \( v_1 \) to \( v_i \neq v_1 \) meeting all \( v_j \) (possibly as an end-vertex). Similarly for \( C' \). Thus there is an \( i > 1 \) such that this holds for both \( v_i \) and \( v'_i \) with arcs \( P \) and \( P' \) respectively. The cycle \( v_1Pv_iP'v'_1v_1 \) has two chords \( v_kv'_k, k \in \{2, 3, 4\} \setminus \{i\} \).

\[ \text{Lemma 8.} \quad P_s \cup C_s + 5e \supseteq C_s^{+2}. \]

\[ \text{Proof.} \] Fix an orientation of \( P \) and denote the edges between \( P = P_s \) and \( C = C_s \) as \( e_i = u_iv_i, i = 1, \ldots, 5 \), where \( u_i \in P \) are ordered by their location along \( P \). (If two edges are incident to the same vertex on \( P \) we order them arbitrarily.) Suppose first that \( v_1 \neq v_5 \). Then one of the arcs in \( C \) from \( v_1 \) to \( v_5 \) in \( C \) must meet two of the three remaining \( v_i \). Then the cycle \( u_1v_1 \ldots u_5 \) contains at least two chords. Thus we may assume \( v_1 = v_5 \). Repeating this argument with \( v_4 \) in place of \( v_5 \) we are also done unless \( v_4 = v_1 \) or the cyclic ordering of the vertices on \( C \) is \( v_1 = v_5, v_2, v_4, v_3 \) with \( v_1, v_2, v_3, v_4 \) distinct. Similarly, using \( v_2 \) in place of \( v_1 \) we are done unless \( v_2 = v_5 \) or the cyclic ordering of the vertices on \( C \) is \( v_1 = v_5, v_4, v_3, v_2 \) and \( v_1, v_2, v_3, v_4 \) are distinct. Since these cyclic orderings are incompatible, we may assume either \( v_4 = v_1 \) or \( v_2 = v_5 \), say \( v_4 = v_1 \). But then \( v_1, v_2, v_3, v_4 \) are not distinct, so \( v_2 = v_5 \) also, and so \( v_1 = v_2 = v_4 = v_5 \). But then the cycle with edges \( e_1 \) and \( e_3 \) contains chords \( e_2 \) and \( e_4 \).

\[ \text{Lemma 9.} \quad P_n \cup P_s + k_ne \supseteq C_s^1 \text{ and } P_n \cup P_s + k'_ne \supseteq C_s^1 \] where \( k_n \) and \( k'_n \) are given by the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>( \geq 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_n )</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( k'_n )</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

In particular \( P_n \cup P_s + (n + 4)e \supseteq C_s^1 \).

\[ \text{Proof.} \] We first show that \( P_s \cup P_s + 8e \supseteq C_s^{+2} \). Let the 8 edges between the paths be \( e_i = u_iv_i \) with \( u_i \) on the first path \( P \) and \( v_i \) on the second path \( P' \).
Let $e_1$ and $e_2$ be the edges whose end-vertices $u_1$ and $u_2$ are farthest apart on $P$. The end-vertices $v_1$ and $v_2$ in general cut the other path $P'$ into three pieces $A'$, $B'$, and $C'$, where $B'$ is the path from $v_1$ to $v_2$ inclusive. If there is no $C_{e_2}^{+2}$ subgraph, then there can be at most one edge from $B'$ to $P$, and by Lemma 8 there can be at most 4 edges from either $A'$ or $C'$ to $P$ (see Figure 3(a)). As there are 8 edges in total, there must be at least one edge to both $A'$ and $C'$. Working off $P'$ we also have edges $e_3$ and $e_4$ where $v_3 \in A'$ and $v_4 \in C'$ are farthest apart on $P'$. Ordering the vertices in each path left to right we may now assume $v_3 < v_1, v_2 < v_4$ and $u_1 < u_3, u_4 < u_2$. We may also assume without loss of generality that $u_1, u_2$ and $v_3, v_4$ are the end-vertices of $P$ and $P'$ respectively. If $u_4 < u_3$ and $v_1 < v_2$ then we obtain a cycle through all vertices except those between $u_3$ and $u_4$ and those between $v_1$ and $v_2$ (see Figure 3(b)). However each of these intervals meets at most one edge $e_i, i > 4$, so this cycle has at least $8 - 4 - 2 = 2$ chords. Thus, by reversing one path if necessary and relabeling the edges, we may assume that $u_3 \leq u_4$ and $v_1 \leq v_2$. As with $P'$, decompose $P$ into three paths $A$, $B$, $C$, where $B$ consists of the vertices from $u_3$ to $u_4$ inclusive.

Figure 3: Finding a $C_{e_2}^{+2}$ in $P_6 \cup P_6 + 8e$ in Lemma 9
By considering the cycles $ABC'B'$ and $BCB'A'$ we see that there is at most one edge from $AB$ to $B'C'$ and at most one edge from $BC$ to $A'B'$. Thus there are two edges in total that either go between $A$ and $A'$ or go between $C$ and $C'$. However by considering the cycles $Au_{3}A'v_{3}$ and $Cu_{3}C'v_{2}$ we see that both edges cannot go between the same pair, so there is an edge $e_{5}$ from $A$ to $A'$ and an edge $e_{6}$ from $C$ to $C'$. On $P$ we have $u_{1} \leq u_{5} < u_{3} \leq u_{4} < u_{6} \leq u_{2}$ and on $P'$ we have $v_{3} \leq v_{5} < v_{1} \leq v_{2} < v_{6} \leq v_{4}$ (see Figure 3(c)).

There are two remaining edges, $e_{7}$ from $AB$ to $B'C'$ and $e_{8}$ from $BC$ to $A'B'$. Suppose that either $u_{7} \in B$ or $v_{8} \in B'$ or $u_{7}$ and $v_{8}$ lie on the same side of $e_{5}$. Then there is a path from $B$ to $B'$ through $A \cup A'$ meeting both $u_{7}$ and $v_{8}$ (see Figure 3(d)). If a similar situation holds for $u_{8}$ and $v_{7}$ on the other side of the graph then we can combine these paths to obtain a cycle through $B$ and $B'$ with $e_{7}$ and $e_{8}$ as chords. Thus without loss of generality we may assume that $u_{1} \leq u_{7} < u_{5}$ and $v_{5} < v_{8} < v_{1}$.

We may assume $v_{7} > v_{1}$ and $u_{8} > u_{3}$ as otherwise there would be a cycle through $e_{1}$ and $e_{3}$ with two chords. If $v_{7} \geq v_{6}$ and $u_{8} \leq u_{6}$ then there is a cycle with chords $e_{5}$ and $e_{6}$ as shown in Figure 3(e). If $v_{7} \geq v_{6}$ and $u_{8} \geq u_{6}$ then there is a cycle with chords $e_{5}$ and $e_{8}$ as shown in Figure 3(f). Finally, if $v_{7} < v_{6}$ then there is a cycle with chords $e_{5}$ and $e_{7}$ as shown in Figure 3(g).

If the $C_{s}^{+}$ obtained uses all the vertices of both paths, then it forms a cycle with $(|P| - 1) + (|P'| - 1) + 8 - (|P| + |P'|) = 6$ chords. Thus by Lemma 5 there is a smaller $C_{s}^{1}$ as a subgraph. Hence $P_{s} \cup P_{s} + 8e \supset C_{s}^{1}$.

To prove the results of the form $P_{n} \cup P_{m} + ke \supset C_{s}^{1}$ for $n < 8$ and all $m$, it is enough to consider $m \leq 2k - 1$ with the edges between $P_{n}$ and $P_{m}$ meeting the first and last vertices of $P_{m}$ and at least one of any pair of adjacent vertices on $P_{m} = v_{1} \ldots v_{m}$. This is because any counterexample with the $k$ added edges missing vertices $v_{i}$ and $v_{i+1}$, or missing $v_{1}$ or $v_{m}$, can be converted by the removal of a vertex of $P_{m}$ into a counterexample with the path $P_{m}$ replaced with $P_{m-1}$. Hence the remaining cases reduce to a finite case analysis. These cases were checked by computer yielding the stated results (see [9]).

**Lemma 10.** For $n \geq 5$ and $m \leq n$, $C_{n} \cup C_{m} + (2n + 1)e \supset C_{s} \cup C_{s}^{1}$. Also $C_{4} \cup C_{4} + 11e \supset C_{3} \cup C_{4}^{+}$, $C_{4} \cup C_{4} + 9e \supset C_{s} \cup C_{s}$, $C_{4} \cup C_{3} + 9e \supset C_{3} \cup C_{3}$.

**Proof.** Drop the condition that $m \leq n$ and assume we have $2 \max(n, m) + 1$ edges between $C_{n}$ and $C_{m}$ and $\max(n, m) \geq 5$. For $\max(n, m) < 8$, the
minimum number of edges needed to give a $C_\ast$ and a $C_{\ast}^4$ with fewer total vertices was found by exhaustive computer search (see [9]). The results are given in Table 2. Hence from now on we shall assume $\max(n, m) \geq 8$.

Let $v$ be a vertex meeting the maximal number of edges between the two cycles and assume without loss of generality that $v \in C_n$ with $v$ adjacent to $d$ vertices in $C_m$. As the total number of edges between the cycles is more than $2n$, we have $3 \leq d \leq m$.

**Case 1:** $d \geq 6$.

Divide $C_m$ into three vertex disjoint arcs $P_1, P_2, \text{ and } P_3$, so that each $P_i$ contains at least two neighbors of $v$. If there are at least 3 edges from $P_i$ to $P = C_n - v$ then one obtains a $C_\ast$ on $P_i \cup P$ not using all these vertices. (Two edges between two paths are enough to form a cycle, and if this cycle is hamiltonian then a third edge forms a chord, and hence one can find a shorter cycle.) However, there are at least 4 edges from $v$ to $C_m - P_i$ forming a $C_{\ast}^{+2}$ disjoint from this $P_i \cup P$. Hence we may assume each $P_i$ sends at most 2 edges to $C_n - v$. The total number of edges between the cycles is then at most $3 \times 2 + d \leq m + 6$, which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$, a contradiction.

**Case 2:** $d = 5$.

Divide $C_m$ into five vertex disjoint arcs $P_1, \ldots, P_5$, so that each $P_i$ contains one neighbor of $v$. Since there are 4 edges to each $C_m - P_i$, we can as in Case 1 assume there are at most 2 edges from each $P_i$ to $C_n - v$. This gives at most $5 \times 2 + d = 15$ edges between the cycles, which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$.

**Case 3:** $d = 4$.

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<th>$n \backslash m$</th>
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<td>13</td>
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</tbody>
</table>

Also, $C_4 \cup C_4 + 9e \supseteq C_\ast \cup C_{\ast}$, $C_4 \cup C_3 + 9e \supseteq C_3 \cup C_3$. 

Table 2: Minimum $k$ such that $C_n \cup C_m + ke \supseteq C_\ast \cup C_{\ast}^4$. 

case 1: $d \geq 6$.

divide $c_m$ into three vertex disjoint arcs $p_1, p_2, \text{ and } p_3$, so that each $p_i$ contains at least two neighbors of $v$. if there are at least 3 edges from $p_i$ to $p = c_n - v$ then one obtains a $c_\ast$ on $p_i \cup p$ not using all these vertices. (two edges between two paths are enough to form a cycle, and if this cycle is hamiltonian then a third edge forms a chord, and hence one can find a shorter cycle.) however, there are at least 4 edges from $v$ to $c_m - p_i$ forming a $c_{\ast}^{+2}$ disjoint from this $p_i \cup p$. hence we may assume each $p_i$ sends at most 2 edges to $c_n - v$. the total number of edges between the cycles is then at most $3 \times 2 + d \leq m + 6$, which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$, a contradiction.

case 2: $d = 5$.

divide $c_m$ into five vertex disjoint arcs $p_1, \ldots, p_5$, so that each $p_i$ contains one neighbor of $v$. since there are 4 edges to each $c_m - p_i$, we can as in case 1 assume there are at most 2 edges from each $p_i$ to $c_n - v$. this gives at most $5 \times 2 + d = 15$ edges between the cycles, which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$.

case 3: $d = 4$.
Divide $C_m$ into two vertex disjoint paths $P_1, P_2$, each containing two neighbors of $v$. Each $P_i$ forms a cycle with $v$. If there are $|P_j| + 4$ edges from $P_j$ to $P = C_n - v$, $j \neq i$, then by Lemma 9 we obtain a $C^*_i$ on these vertices which does not use all the vertices of $P_j \cup P$ and is disjoint from a cycle in $P_i \cup \{v\}$. Thus we may assume there are at most $|P_j| + 3$ edges from each $P_j$ to $P$. The total number of edges between the cycles is then at most $\sum_{i=1}^{2}(|P_i| + 3) + d = m + 10$. If $m > 6$ then we can in fact choose the paths $P_i$ so that the end-vertices of each $P_i$ are not both neighbors of $v$ and so the cycle in $P_i \cup \{v\}$ does not use all the vertices of $P_i$. Then by Lemma 9 we can assume there are at most $|P_j| + 2$ edges from each $P_j$ to $P$ and we obtain a bound of $\sum_{i=1}^{2}(|P_i| + 2) + d = m + 8$ on the number of edges between the cycles. Thus in general there are at most $\max(m + 8, 16)$ edges between the cycles which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$.

**Case 4:** $d = 3$.

Let $v_1, v_2, v_3$ be the neighbors of $v$ on $C_m$, and let $P_i$ be the arc strictly between $v_i$ and $v_{i+1}$ (where $v_4 = v_1$). Let $p_i$ be the number of edges from $P_i$ to $P = C_n - v$ and $n_i$ the number of edges from $v_i$ to $P$. We can form a cycle $vv_1P_1v_2v$, so if there are more than 7 edges from $P_2v_3P_3$ to $P$ then we are done by Lemma 9. Hence we may assume $p_2 + p_3 + n_3 \leq 7$. Adding the three cyclic rearrangements of this inequality gives $2\sum p_i + \sum n_i \leq 21$. But as the maximum number of edges meeting a vertex is $d = 3$, we have $n_i \leq 2$. (We are not counting the edge from $v_i$ to $v$ in $n_i$.) Hence $2\sum p_i + 2\sum n_i \leq 21 + 6 = 27$, so $\sum p_i + \sum n_i \leq 13$. Hence there are at most $13 + d = 16$ edges between the cycles, which is less than $2 \max(n, m) + 1$ when $\max(n, m) \geq 8$. The lemma now follows.

**Lemma 11.** For $n \geq 6$ and $4 \leq m \leq n$, $C_n \cup C_m + (3n + 1)e \supset C^*_n \cup C^*_m$.

**Proof.** Drop the condition that $m \leq n$ and assume we have $3 \max(n, m) + 1$ edges between $C_n$ and $C_m$ and $\max(n, m) \geq 6$. For the cases when $\max(n, m) \leq 7$ the minimum number of edges needed to give two vertex disjoint $C^*_i$ graphs with fewer total number of vertices was found by exhaustive computer search (see [9]). The results are given in Table 3. Hence from now on we shall assume $\max(n, m) \geq 8$.

Let $v, v'$ be a pair of vertices that are adjacent on one of the cycles and which between them meet the maximum number of edges to the other cycle. Assume without loss of generality that $v, v' \in C_n$ with $d_2$ edges between
Table 3: Minimum $k$ such that $C_n \cup C_m + ke \supset C^*_n \cup C^*_m$.

<table>
<thead>
<tr>
<th>$n \setminus m$</th>
<th>4</th>
<th>5</th>
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<td>7</td>
<td>15</td>
<td>17</td>
<td>17</td>
<td>18</td>
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</table>

{$v, v'$} and $C_m$. As the total number of edges between the cycles is more than $3n$, we have $d_2 \geq 7$. A vertex $u \in C_m$ will be said to have multiplicity $k$, $k \in \{0, 1, 2\}$, if it is joined to $k$ of the elements of {$v, v'$}. Let $n_k$ be the number of multiplicity $k$ vertices of $C_m$ so that $m = n_0 + n_1 + n_2$ and $d_2 = n_1 + 2n_2$. Let $P$ be an arc of $C_m$ meeting some neighbors of {$v, v'$}. If the multiplicities of the neighbors along $P$ contains one of the patterns 21, 2.2, 2..1., 2..1., 1..1..1..1, (1)
(or their reflections) where .. denotes zero or more 0s, then there is a $C^*_n$ in the graph using the vertices of {$v, v'$} $\cup P$. (The last two cases follow from Lemma 9 with $n = 2$, the others are easy exercises.)

**Case 1:** $d_2 > 10$.
Then either $v$ or $v'$ sends at least 6 edges to $C_m$, say $v$ sends $d \geq 6$ such edges. Divide $C_m$ into two (if $d \geq 8$) or three (if $d \in \{6, 7\}$) vertex disjoint arcs $P_i$ so that each $P_i$ meets at most $d - 4$ neighbors of $v$. If there are at least $|P_i| + 4$ edges from $P_i$ to $P = C_n - v$ then by Lemma 9 one obtains a $C^*_n$ on $P_i \cup P$ not using all these vertices. However, there are at least 4 edges from $v$ to $C_m - P_i$ forming a $C^*_n$ disjoint from this $C^*_m$. Hence we may assume each $P_i$ sends at most $|P_i| + 3$ edges to $C_n - v$. For $d \geq 8$ the total number of edges between the cycles is then at most $\sum_{i=1}^{2}(|P_i| + 3) + d = m + 6 + d \leq 2m + 6$ and for $d < 8$ the total number of edges between the cycles is at most $\sum_{i=1}^{3}(|P_i| + 3) + d = m + 9 + d \leq m + 16$. In both cases this is less that $3\max(n, m) + 1$ as $\max(n, m) \geq 8$, contradicting the assumption that there are $3\max(n, m) + 1$ edges between the cycles.

**Case 2:** $d_2 = 10$.
Assume first that one can divide $C_m$ into two vertex disjoint arcs $P_1$ and $P_2$, so that each $P_i$ meets at most $d_2 - 5 = 5$ edges from {$v, v'$}. If there are at least $|P_i| + 4$ edges from $P_i$ to $P = C_n - \{v, v'\}$ then by Lemma 9 one obtains a $C^*_n$ on $P_i \cup P$ not using all these vertices and a $C^*_n$ on the disjoint subset
of vertices of the paths $vv'$ and $C_m - P_i$. Thus the total number of edges between the cycles is at most $\sum_{i=1}^{2}(|P_i| + 3) + d_2 = m + 16$, which is less that $3 \max(n, m) + 1$ as $\max(n, m) \geq 8$. If it is not possible to find two such arcs, then the multiplicity pattern of points on $C_m$ is $2..2.2..2...$. In this case we can still decompose $C_m$ into two arcs $P_i$, each of whose complements includes the pattern $2..2$ of (1). Thus we have a $C_i^1$ on the vertices of $C_m - P_i$ and $\{v, v'\}$, and so the above argument still applies with this pair $(P_1, P_2)$.

**Case 3:** $d_2 = 9$.
Assume first that $n_2 > 0$ and the multiplicity pattern of vertices along $C_m$ is not $2..1..2.1..2..1..$ with at least one multiplicity 0 vertex between consecutive neighbors of $\{v, v'\}$. Then we can find an arc $P_1$ of $C_m$ with multiplicity pattern $21, 2..1..1..1..$, or $2..2$ so that $P_2 = C_m - P_1$ sends at least 5 edges to $\{v, v'\}$. Thus, for $i = 1, 2$, there exists a $C_i^1$ on the vertices of $C_m - P_i$ and $\{v, v'\}$. As in Case 2, we can now assume there are at most $|P_i| + 3$ edges from $P_i$ to $P = C_m - \{v, v'\}$ and so there are at most $\sum_{i=1}^{2}(|P_i| + 3) + d_2 = m + 15$ edges in total between the cycles. This gives a contradiction as $m + 15 < 3 \max(n, m) + 1$ when $\max(n, m) \geq 8$. Assume now that we are in one of the remaining cases where either we have the pattern $2..1..2.1..2..1..$ with at least one multiplicity 0 vertex between consecutive neighbors of $\{v, v'\}$, or $n_2 = 0$. In both cases there are at least 9 vertices in $C_m$ and we can decompose $C_m$ into three vertex disjoint arcs $P_i$ each sending 3 edges to $\{v, v'\}$. Then $C_m - P_i$ sends 6 edges to $\{v, v'\}$ and as above we can assume there are at most $\sum_{i=1}^{3}(|P_i| + 3) + d_2 = m + 18$ edges between the cycles. This gives a contradiction as $m + 18 < 3 \max(n, m) + 1$ when $\max(n, m) \geq m \geq 9$.

**Case 4:** $d_2 = 8$.
If the multiplicity pattern on $C_m$ is one of the following,

$$2..2..2.., \quad 2..2..1..1.., \quad 2..2..1..1..1.., \quad 2..1..2..1..1..,$$

or if we have adjacent vertices with multiplicities 1 and 2, then we can decompose $C_m$ into two arcs $P_1, P_2$, each containing one of the multiplicity patterns in (1). Thus each $C_m - P_i$ forms a $C_i^1$ with $\{v, v'\}$. As above we get a bound of $\sum_{i=1}^{2}(|P_i| + 3) + d_2 = m + 14$ on the number of edges between the cycles. This gives a contradiction as $m + 14 < 3 \max(n, m) + 1$ when $\max(n, m) \geq 8$. The remaining cases when $n_2 > 0$ are (up to cyclic rearrangements)

$$2..2..1..1.., \quad 2..1..2..1..1.., \quad 2..1..1..1..1..,$$

with multiplicity 0 vertices between the 1s and 2s. In each of these cases there are at least 9 vertices on $C_m$ so $m \geq 9$. Also, in each case (and when
we can decompose \(C_m\) into three arcs \(P_i\) each meeting at most 3 edges to \(\{v, v'\}\). Then \(C_m - P_i\) sends at least 5 edges to \(\{v, v'\}\), so as above we can assume there are at most \(\sum_{i=1}^{3} (|P_i| + 3) + d_2 = m + 17\) edges between the cycles. This is less than \(3 \max(n, m) + 1\) if \(\max(n, m) \geq 9\), so we obtain a contradiction unless \(n \leq m = 8\) and \(n_2 = 0\). In this last case we may assume that each \(P_i\) contains either 2 or 3 vertices of \(C_m\) as there are no multiplicity 0 vertices on \(C_m\). By Lemma 9 we can in fact assume each \(P_i\) sends at most 5 edges to \(C_{n - \{v, v'\}}\) so our bound on the number of edges between the cycles is \(3 \times 5 + d_2 = 23\) which is less than \(3 \max(n, m) + 1 = 25\).

**Case 5:** \(d_2 = 7\).

If \(n_2 > 0\) then the multiplicity pattern on \(C_m\) is one of the following (up to cyclic rearrangements).

\[
2..2..1., \quad 2..2..1..1., \quad 2..1..2..1.., \quad 2..1..1..1..1. 
\]

In each case we can decompose \(C_m\) into three arcs \(P_i\) such that \(C_m - P_i\) always contains one of the patterns of (1). As above the total number of edges between the cycles is at most \(\sum_{i=1}^{7} (|P_i| + 3) + d_2 = m + 16\) which is less than \(3 \max(n, m) + 1\) when \(\max(n, m) \geq 8\). Hence we may assume \(n_2 = 0\).

Let \(v_1, \ldots, v_7\) be the neighbors of \(\{v, v'\}\) on \(C_m\) arranged in a cyclic order, and let \(P_i\) be the arc of \(C_m\) from \(v_i\) through \(v_{i+1}\) to just before \(v_{i+2}\) (indices taken mod 7). Thus \(P_1, \ldots, P_7\) form a double cover of \(C_m\). Each \(C_m - P_i\) sends 5 edges to \(\{v, v'\}\) so as above we may assume each \(P_i\) sends at most \(|P_i| + 3\) edges to \(C_n - \{v, v'\}\). Thus the total number of edges between the cycles is at most \(\frac{1}{2} \sum_{i=1}^{7} (|P_i| + 3) + d_2 = m + 17\frac{1}{2}\). If \(\max(n, m) \geq 9\) we are done as this is less than \(3 \max(n, m) + 1\). Hence we may now assume \(\max(n, m) = 8\). In this case there is at most one multiplicity 0 vertex on \(C_m\) so all \(P_i\) have either 2 or 3 vertices. Then by Lemma 9 we may assume that there are at most 5 edges from \(P_i\) to \(C_n - \{v, v'\}\) and we obtain a bound of \(\frac{1}{2} \sum_{i=1}^{7} 5 + d_2 = 24\frac{1}{2}\) on the total number of edges between the cycles. This is a contradiction as \(3 \max(n, m) + 1 = 25\) in this case.

The lemma now follows as \(d_2 \geq 7\).

**Theorem 12.** Suppose \(G\) is a graph with minimum degree \(\delta(G) \geq 2\). Suppose further that if there is more than one vertex of degree 2 in \(G\) then the degree 2 vertices of \(G\) induce a path in \(G\). Then \(G\) contains a cycle with at least two non-incident chords.
Proof. By considering any single component of \( G \), we may assume without loss of generality that \( G \) is connected. Let \( P_0 \) be the subgraph of \( G \) induced by the degree 2 vertices, so that \( P_0 \) is either a path, a single vertex, or empty. Suppose \( G \) is not 2-connected and let \( B_1, \ldots, B_r \) be the blocks in the block cut-vertex decomposition of \( G \). Further, suppose that if \( P_0 \neq \emptyset \) then it meets \( B_1 \). Take any leaf-block \( B_i \neq B_1 \). Then \( B_i \) is not a single edge (as \( G \) has no degree 1 vertices), and can meet \( P_0 \) in at most one vertex (the cut vertex joining \( B_i \) to the rest of the graph). Hence \( B_i \) is 2-connected and all vertices in \( B_i \) have degree at least 3 except for the cut vertex joining \( B_i \) to the rest of \( G \), which has degree at least 2 in \( B_i \). By replacing \( G \) with \( B_i \) we may therefore assume \( G \) is 2-connected.

Since \( G \) is non-trivial and 2-connected, it contains a cycle. Pick a longest cycle \( C \) in \( G \). The graph \( G \setminus C \) is a union of components \( S_1, \ldots, S_r \). For each chord \( uv \) of \( C \) we also include a fictitious empty component \( S_i \) that we declare to be joined to \( u \) and \( v \) only. In this way, each vertex \( v \in C \), \( v \notin P_0 \), must be joined to at least one \( S_i \) as \( d(v) \geq 3 \). If two neighboring vertices \( u, v \) on \( C \) are joined to the same \( S_i \) then we can construct a longer cycle by replacing the edge \( uv \) of \( C \) by a path through \( S_i \) (which in this case is necessarily non-empty). However, each \( S_i \) must be joined to at least two vertices of \( C \) as \( G \) is 2-connected. We shall construct a new cycle with two non-incident chords using paths \( P_i \) joining vertices of \( C \) through the \( S_i \), and edges of \( C \). The chords will themselves be original edges of \( C \).

For the rest of the proof we shall drop the assumption on the internal structure of the \( S_i \), and use only the fact that two vertices on \( C \) joined to the same \( S_i \) can be joined by a path through \( S_i \). Also, all vertices on \( C \) are joined to some \( S_i \), except possibly those on a proper arc \( P_0 \) of \( C \), and each \( S_i \) is joined to at least two vertices of \( C \), no pair of which are adjacent on \( C \). This slight generalization will be used in the proof of Theorem 13 below. Fix an orientation of \( C \) and for \( u, v \in C \) write \([u, v]\) for the arc from \( u \) to \( v \) clockwise around \( C \) including the endpoints \( u \) and \( v \). We shall also write \((u, v), [u, v], (u, v)\) for arcs that do not include endpoints \( u, v \), or both \( u \) and \( v \) respectively. For \( x \in C \) we shall abuse notation slightly by writing \( S_x \) for some \( S_i \) that is joined to \( x \), even though the choice of \( S_i \) may not be
unique.

Pick two vertices $a$ and $a'$ that are joined to a common $S_a$ and with minimal distance between $a$ and $a'$ in the path $C - P_0$. The assumptions above guarantee such a pair exists and that they are not adjacent on $C$. Let $[a, a'] = ab\ldots ca'$ be the arc from $a$ to $a'$ in $C - P_0$. Assume first that $b \neq c$. By minimality of $[a, a']$, we may assume $b$ and $c$ are joined to distinct sets $S_b$ and $S_c$, neither of which is $S_a$ as $S_a$ is not joined to neighboring vertices on $C$. Let $b' \neq b$ and $c' \neq c$ be vertices of $C$ joined to $S_b$ and $S_c$ respectively, and for $x \in \{a, b, c\}$ let $P_x$ be a path through $S_x$ joining $x$ to its corresponding $x'$. By minimality of $[a, a']$, $b'$ and $c'$ do not lie in the arc $[a, a']$. If $b' \neq c'$ then there is a cycle with chords $ab$ and $ca'$ (see Figure 4(a) and (b)). If $b' = c'$ and this vertex is adjacent to $a$ or $a'$, say $a'$, then by Figure 4(d) we obtain a cycle with two chords $ab$ and $a'b'$. Thus we may assume $b' = c'$ and this vertex is not a neighbor of $a$ or $a'$ on $C$. Also, we can assume that neither $S_b$ nor $S_c$ is joined to any vertex not in $\{b, c, b'\}$, so in particular they are not joined to any vertex in $[a', b']$. If $b = c$ then we have the situation in Figure 4(e), where once again we may assume any other vertex $b' = c'$ joined to $S_b$ is not a neighbor of $a$ or $a'$ and, by suitable choice of $b' = c'$, we can again assume $S_b = S_c$ is not joined to any vertex in $[a', b']$. In the following we shall consider the case when $b \neq c$ and will only mention the $b = c$ case when there are differences in the proof.

As $[a, a']$ is disjoint from $P_0$, we may assume without loss of generality that $P_0$ (if non-empty) lies in $(b', a)$ and hence that the vertex $d \in [a', b')$ adjacent to $a'$ is not of degree 2 and thus is joined to some $S_d$. Now $S_d \neq S_a$ as otherwise $S_a$ would be joined to neighboring vertices $a', d$ on $C$, and $S_d \neq S_b, S_c$ as both $S_b$ and $S_c$ are not joined to any vertex in $[a', b')$, which includes the vertex $d$. Let $d' \neq d$ be a vertex on $C$ joined to $S_d$. Then $d'$ does not lie in the arc $(b, a')$ by minimality of $[a, a']$. Also $d' \neq b$ as otherwise we could use
{b, d} in place of \{b, b'\} to obtain Figure 4(d). If \(d' \in (b', a)\) then we have a cycle with chords \(ab\) and \(a'd\) (Figure 5(a)). Hence we may assume \(d' \in [d, b']\). Now choose \(e, e' \in [d, b']\) joined to the same \(S_e \notin \{S_a, S_b, S_c\}\) with minimal arc-length \([e, e']\). As \(d, d'\) have the required properties that \(d, d' \in [d, b']\) and \(S_d \notin \{S_a, S_b, S_c\}\), such a pair \(e, e'\) must exist. From now on we shall ignore \(\{d, d'\}\) and work instead with \(\{e, e'\}\), although these may be the same pair.

Let \([e, e'] = ef \ldots g e'\), possibly with \(f = g\). Note \([e, e']\) is disjoint from \(P_0\), so \(f\) and \(g\) are joined to some \(S_f\) and \(S_g\) respectively. We have \(S_f, S_g \notin \{S_b, S_c\}\) as \(S_b\) and \(S_c\) are not joined to any vertex in \([a', b')\), which includes both \(f\) and \(g\). Also \(S_f, S_g \neq S_e\) as otherwise \(S_e\) would be joined to adjacent vertices of \(C\). Now both \(f\) and \(g\) are joined via \(S_f\) and \(S_g\) to vertices \(f'\) and \(g'\) respectively outside of \([e, e']\) for otherwise \(f' \in [e, e']\) or \(g' \in [e, e']\) would contradict the minimality of \([e, e']\). (If \(S_f = S_a\), say, then this is automatic as we can take \(f' = a\).) If precisely one of \(S_f\) and \(S_g\) are equal to \(S_a\), say \(S_f = S_a\), then we can take \(f' = a\) or \(f' = a'\) so that \(f' \neq g'\). Thus we are in the case of Figure 4(a) or (b) with \((\{e, e'\}, \{f, f'\}, \{g, g'\})\) in place of \((\{a, a'\}, \{b, b'\}, \{c, c'\})\). If \(S_f = S_g = S_a\) and \(e' \neq b'\) then we are in the case of Figure 4(a) with \((\{a, f\}, \{b, b'\}, \{e, e'\})\) in place of \((\{a, a'\}, \{b, b'\}, \{c, c'\})\). If \(S_f = S_g = S_a\) and \(e' = b'\) then we are in the case of Figure 4(d) with \(\{a, g\}\) taking the place of \(\{a, a'\}\). Hence we may assume \(S_f, S_g \notin \{S_a, S_b, S_c, S_e\}\).

If \(f \neq g\) then \(S_f \neq S_g\) (by minimality of \([e, e']\)) and so arguing as above with \(\{e, e'\}\) in place of \(\{a, a'\}\) we can assume \(f' = g'\). If \(f = g\) then we also choose \(f' = g'\). If \(f' \in [a', e]\) then we have a cycle with chords \(ca'\) and \(e'g\)
Figure 5(c). If $f' \in (e', b']$ then we have a cycle with chords $ca'$ and $ef$ (Figure 5(d), note $a' \neq e$ so chords do not intersect). If $f' \in (b', a]$ then we have a cycle with chords $ab$ and $e'g$ (Figure 5(e)). If $f' \in (a, a')$ and $e' \neq b'$ then we have a cycle with chords $ab$ and $ef$ (Figure 5(f)). Finally, if $f' \in (a, a')$ and $e' = b'$ then we have a cycle with chords $ab$ and $e'g$ (Figure 5(g)).

Theorem 13. Suppose $G$ contains a cycle $C$ and every vertex in $S = G - C$ has degree at least 3 in $G$ and $S \neq \emptyset$. Then either $G$ contains a cycle shorter than $C$, or $G$ contains a cycle with two chords.

Proof. By restricting to a single component of $G[S]$ we may assume $G[S]$ is connected. If $S$ consists of a single vertex then it sends at least three edges to $C$, so we are done by Lemma 3. Hence we may assume $S$ contains at least two vertices. Consider the block cut-vertex decomposition of $G[S]$. Suppose there is a leaf-block $B$, possibly joined to the rest of $G[S]$ via a cut-vertex $v_1$ and suppose that there are no edges from $B - v_1$ to $C$ in $G$. Then every vertex in $B$ except possibly $v_1$ is of degree at least 3 in $G[B]$, so by Theorem 12 we have a $C^+_k$ in $G[B]$. Hence we may assume that there is an edge from $B - v_1$ to $C$.

Each leaf block is either a single edge or is 2-connected. Suppose first that there exists a 2-connected leaf-block $B$. If $B \neq S$ let $v_1$ be the cut-vertex joining $B$ to the rest of $S$ in $G[S]$. Since every other leaf-block is joined to $C$, we may assume there is a path $P$ from $v_1$ to $C$ that does not meet $B - v_1$. If $B = S$ then we can set $v_1$ to be any vertex of $B$ joined to $C$ and $P$ to be the single edge path joining $v_1$ to $C$. Then in the graph $G[B \cup P \cup C]$, each vertex of $B$ has degree at least 3.

Pick a maximal cycle $C'$ in $G[B]$. Then as in the proof of Theorem 12, we can decompose $G[B] - C'$ into components $S_i$. If $(P - v_1) \cup C$ is adjacent to vertices in $C''$ we shall consider this an extra component, which will be denoted $S_1$. Chords of $C''$ will be associated to fictitious empty components $S_i$. Now each vertex of $C'$ is joined to some $S_i$, and each $S_i \neq S_1$ is joined to at least two vertices of $C''$. Moreover, each $S_i \neq S_1$ cannot be joined to adjacent vertices on $C''$ by maximality of $C'$. The component $S_1$ however may be joined to any number of vertices of $C'$ and even to adjacent vertices of $C'$ as the cycle $C'$ was chosen to be maximal in $G[B]$ rather than in $G[B \cup P \cup C]$. By removing any edges to $S_1$ from each $v \in C'$ that is joined to some $S_i \neq S_1$, we may
assume that any vertex $v \in C'$ that is joined to $S_1$ is joined to no other $S_i$. If $S_1$ is now joined only to the vertices of $C'$ forming an proper arc of $C'$, then we denote this arc as $P_0$, remove all connections to $S_1$ and proceed with the proof as in Theorem 12. (This includes the case when $S_1$ is joined to zero or one vertex of $C'$, but not the case when $S_1$ is joined to all vertices of $C$, which will be dealt with below.) If the set of vertices joined to $S_1$ forms just one arc plus some other isolated vertices on $C'$, then we let $P_0$ contain all but one end-vertex of the arc. Remove connections from $P_0$ to $S_1$. Then $S_1$ is now joined to at least two vertices of $C'$, but is not joined to any pair of adjacent vertices on $C'$. In this case we can proceed as in Theorem 12. Now suppose there are two non-trivial arcs of vertices joined to $S_1$. Then in particular there are at least 4 vertices of $C'$ joining to $S_1$. Thus we may assume that $C$ and $C'$ are joined by 3 edges and a path $P$ (which may itself be a single edge), and these meet two pairs of adjacent vertices on $C'$. The only other case is when $S_1$ is joined to all vertices of $C'$. In both these cases, let $v_1P_1u_1, v_2u_2, \ldots, v_ku_k, k \in \{3, 4\}$ be paths and edges from $C'$ to $C$ with $v_1$ adjacent to $v_2$, and $v_3$ either adjacent to $v_4$ (if $k = 4$) or adjacent to both $v_1$ and $v_2$ (if $k = 3$). The last case is needed only if $C'$ is a triangle and $S_1$ is joined to all vertices of $C'$. If $u_1 \neq u_2$ then one of the arcs from $u_1$ to $u_2$ on $C$ contains $u_3$ and we obtain a cycle $v_1 \ldots v_3 \ldots v_2 v_2 \ldots u_3 \ldots u_1 P v_1$ with chords $v_1 v_2$ and $v_3 u_3$. Thus we may assume $u_1 = u_2$. Similarly if $k = 4$ we may assume $u_3 = u_4$ (as otherwise we would obtain a cycle with chords $v_3 v_4$ and $v_2 u_2$). If $k = 3$ we may assume (by interchanging $v_2, u_2$ and $v_3, u_3$) that $u_1 = u_2 = u_3$. If $u_1 = u_2 = u_3 = u$ then we have a cycle $v_1 P u_1 v_2 \ldots v_3 \ldots v_1$ with chords $v_1 v_2$ and $u_3 v_3$ (see Figure 6(a)). If $u_1 = u_2 \neq u_3 = u_4$ then we have a cycle $u_1 P v_1 \ldots v_3 u_4 v_3 \ldots v_2 u_1$ with chords $v_1 v_2$ and $v_3 v_4$ (Figure 6(b)).

Now suppose there are no 2-connected leaf blocks of $G[S]$. Then either $G[S]$ is a single edge, or its block cut-vertex decomposition contains at least two leaf blocks which are single edges. In either case there is a path $P$ joining two

Figure 6: Proof of Theorem 13.
vertices $v_1$ and $v_2$ of degree 1 in $G[S]$. Since $v_1$ and $v_2$ have degree at least 3 in $G$, each must be joined to two vertices of $C$. Denote the neighbors of \{v_1, v_2\} in $C$ as $u_1, \ldots, u_k$, where $2 \leq k \leq 4$ and the $u_i$ are arranged cyclically around $C$. There must be some consecutive pair, say $u_1, u_2$ such that $u_1$ is joined to $v_1$ and $u_2$ is joined to $v_2$. Then $v_1Pv_2u_2\ldots u_3\ldots u_k\ldots u_1v_1$ is a cycle with two chords $v_1u_i$ and $v_2u_j$ for some $i, j \in \{1, \ldots, k\}$. □

References


[9] Supplemental material including computer code and results can be found at http://memphis.edu/msci/people/pbalistr/cyclechord.php.