Interference percolation

Paul Balister*    Béla Bollobás†‡

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Abstract

Let $G$ be an infinite connected graph with minimum degree $\delta$ and maximum degree $\Delta$. Let $G_p$ be a random induced subgraph of $G$ obtained by selecting each vertex of $G$ independently with probability $p$, $0 < p < 1$, and let $G^\leq p_k$ be the induced subgraph of $G_p$ obtained by deleting all vertices of $G_p$ with degree greater than $k$ in $G_p$. We show that if $\delta \geq 6$ and $\Delta/\delta$ is not too large then $G^\leq p_3$ almost surely has no infinite component. Moreover, this result is essentially best possible since there are examples where $G^\leq p_k$ has an infinite component (a) when $\delta = \Delta = 3, 4, \text{ or } 5$, and $k = 3$; (b) when $\Delta \gg \delta$ for any $\delta$ and $k = 3$; and (c) when $\delta = \Delta$ for any $\delta \geq 3$ and $k \geq 4$. In addition, we show that if $G$ is the $d$-dimensional lattice $\mathbb{Z}^d$ then $G^\leq 1/d$ almost surely has an infinite component for sufficiently large $d$.

1 Introduction

We consider the following problem. Suppose we have a communication network consisting of transceivers (vertices) and communication links (edges) forming a graph $G$. Suppose at some time each transceiver is active independently with probability $p$. We suppose that a message can be relayed through active transceivers, but with the restriction that

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*Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152. E-mail: pbalistr@memphis.edu
†Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, CB3 0WB; and Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152. E-mail: B.Bollobas@dpmms.cam.ac.uk
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if a transceiver has too many active neighbors, then interference between the signals means that this transceiver cannot effectively relay a message. We wish to know whether a message can be propagated effectively. More formally, let $G$ be a countably infinite graph and let $G_p$ be the random induced subgraph of $G$ obtained by selecting each vertex of $G$ independently with probability $p$. Let $G_p^{\leq k}$ be the induced subgraph of $G_p$ obtained by deleting all vertices of $G_p$ with degree greater than $k$. Define

$$P_k(G) = \{ p \in (0, 1) : \mathbb{P}(G_p^{\leq k} \text{ has an infinite component}) > 0 \}.$$ 

If $p \in P_k(G)$ then we say that for site probability $p$ we have $k$-interference percolation on $G$. Note that there is no monotonicity in $p$. Indeed, we expect percolation to fail in general both for very small $p$ (when there is no percolation even without interference) and for very large $p$ (when almost all vertices of $G_p$ are deleted due to interference). We will be mainly interested in whether $P_k(G)$ is empty or not: we say $G$ $k$-percolates if $P_k(G) \neq \emptyset$, i.e., for some $p$ with $0 < p < 1$, $G_p^{\leq k}$ has an infinite component with positive probability. In fact, for locally finite graphs (graphs for which each vertex has finite degree), the event that $G_p^{\leq k}$ has an infinite component is unaffected by changing the state of a finite number of vertices. Thus, by the Kolmogorov 0–1 law, the probability that there is an infinite component is either 0 or 1. This may fail for graphs that are not locally finite. As an example, let $G$ be a locally finite graph that $k$-percolates and form a new graph $G'$ by adding $k + 1$ new vertices to $G$, each joined to all the vertices of $G$. Let $p \in P_k(G)$. Then with probability $p^{k+1}$, all the additional vertices lie in $G_p'$, and so $G_p'^{\leq k}$ is almost surely empty. But with probability $(1 - p)^{k+1}$, none of the additional vertices lie in $G_p'$; in this case $G_p'^{\leq k}$ almost surely has an infinite component. Thus

$$(1 - p)^{k+1} \leq \mathbb{P}(G_p'^{\leq k} \text{ has an infinite component}) \leq 1 - p^{k+1},$$

i.e., the probability that with site probability $p$ we have $k$-interference percolation on $G$ is strictly between 0 and 1. From now on we shall only consider locally finite graphs.

For a graph $G$, write $\delta = \delta(G)$ and $\Delta = \Delta(G)$ for the minimum and maximum degrees of the vertices of $G$ respectively. If the degrees of the vertices of $G$ are unbounded then we set $\Delta(G) = \infty$. Our main result is the following.

**Theorem 1.** Any graph $G$ with $\delta(G) \geq 6$, $\Delta(G) < 2.372(\delta(G) - 3)$, and $(\delta, \Delta) \neq (6, 7)$ does not 3-percolate.

We shall also show that this result is false if we do not bound the maximum degree.

**Theorem 2.** For all $\delta$, there exists a graph with minimum degree $\delta$ and finite maximum degree that 3-percolates.
The maximum degree required for the proof of Theorem 2 is exponential in \( \delta \), while by Theorem 1 we know that \( \Delta \) must be at least a linear multiple of \( \delta \) if \( G \) is to 3-percolate. This suggests the following question.

**Question 1.** What is the smallest value of \( \Delta \), as a function of \( \delta \), such that there exists a 3-percolating graph \( G \) with minimum and maximum degrees \( \delta \) and \( \Delta \) respectively.

Although Theorem 2 shows that in general a bound on \( \Delta \) is required in Theorem 1, there are some cases when Theorem 1 holds without an upper bound on \( \Delta \). One case is when \( G \) is a tree.

**Theorem 3.** If \( T \) is a tree with minimum degree at least 6 then \( T \) does not 3-percolate.

This might seem somewhat surprising since one generally expects percolation on trees to be easier than on general graphs with vertices of the same degree. However, cycles in \( G \) can sometimes make percolation easier when there is interference.

Showing that general graphs do \( k \)-percolate is somewhat harder. However, for trees we prove the following.

**Theorem 4.** The infinite \( d \)-regular tree 3-percolates for \( 3 \leq d \leq 5 \) and 4-percolates for all \( d \geq 3 \). Moreover, for any \( k \geq 4 \) and \( \delta \geq 3 \) there is a \( \Delta = \Delta(k, \delta) \) such that any tree with minimum degree \( \delta \) and maximum degree at most \( \Delta \) \( k \)-percolates, but there exists a tree with minimum degree \( \delta \) and maximum degree \( \Delta + 1 \) that does not \( k \)-percolate.

The function \( \Delta(k, \delta) \) can be calculated fairly easily. Some values are given in Table 1. It is worth noting that for regular \( k \)-percolating trees, the set \( P_k(G) \) of \( p \) for which percolation occurs is an interval. However, this is not true in general, even for trees. Indeed, if \( G \) is obtained by joining two (locally finite) graphs \( G_1 \) and \( G_2 \) at a single vertex then \( P_k(G) = P_k(G_1) \cup P_k(G_2) \). It is easy to find \( G_1 \) and \( G_2 \) for which the closures of \( P_k(G_1) \) and \( P_k(G_2) \) are disjoint, so that \( P_k(G) \) is not an interval.

As another example we show that the (generalized) Gilbert model \[3\] does not 3-percolate.

**Theorem 5.** Let \( G \) be an infinite graph with vertex set given by a uniform Poisson point process in \( \mathbb{R}^d \) for some \( d \geq 2 \), and edges \( uv \) whenever \( u - v \) lies in some fixed bounded symmetric subset \( A \subseteq \mathbb{R}^d \). Then \( G \) does not 3-percolate.

One would expect that this model would \( k \)-percolate for sufficiently large \( k \) and sufficiently large intensity of the Poisson process. Note that if the intensity of the Poisson process is too small, then no percolation occurs even with \( p = 1 \) and without removing high degree vertices. On the other hand, a Poisson process of intensity \( \lambda \) in which
each vertex is kept with probability $p$ is just a Poisson process of intensity $p\lambda$. Thus $k$-percolation with a given $\lambda$ and $p$ implies $k$-percolation with any $\lambda' \geq \lambda$ and $p' := p\lambda/\lambda'$. Hence $k$-percolation is independent of the intensity $\lambda$ provided $\lambda$ is sufficiently large.

In the special case when $A$ is a disk in $\mathbb{R}^2$, numerical simulations suggest that $G$ 7-percolates for large $\lambda$, but does not 6-percolate. It seems likely that this is the worst case.

**Conjecture 1.** Let $G$ be an infinite graph with vertex set given by a uniform Poisson point process in $\mathbb{R}^d$ of intensity $\lambda$ for some $d \geq 2$, and edges $uv$ whenever $u - v$ lies in some fixed bounded symmetric subset $A \subseteq \mathbb{R}^d$ of positive measure. Then $G$ 7-percolates provided $\lambda$ is sufficiently large.

The intuition behind this conjecture is that as one departs from the 2-dimensional disk the graph becomes more tree-like. By the above mentioned independence on the intensity, it should therefore behave more like an almost regular tree of very high degree, which we know 4-percolates. Additional conjectures can be made in higher dimensions. For $d \geq 5$ it appears that even the case when $A$ is a sphere gives 4-percolation, and for $d = 3, 4$, spheres give 5-percolation. Perhaps these are the worst cases in their respective dimensions.

In the case of a disk in $\mathbb{R}^2$ we can give a (rather crude) positive result.

**Theorem 6.** Let $G$ be an infinite graph with vertex set given by a uniform Poisson point process in $\mathbb{R}^2$, and edges $uv$ whenever the Euclidean distance $\|u - v\| \leq R$, where $R$ is any sufficiently large constant (depending on the intensity of the Poisson process). Then $G$ 36-percolates.

Finally, for lattices of high dimension we prove the following precise threshold. Here $\mathbb{Z}^d$ represents the lattice with vertex set $\mathbb{Z}^d$ and edges joining vertices which are at Euclidean distance 1.

**Theorem 7.** For sufficiently large $d$ the lattice $\mathbb{Z}^d$ 4-percolates but does not 3-percolate.

Numerical simulations strongly suggest the following conjecture.

**Conjecture 2.** For all $d \geq 2$ the lattice $\mathbb{Z}^d$ 4-percolates, but does not 3-percolate.

For $d \geq 3$ absence of 3-percolation follows from Theorem 1, but this still leaves the question of 3-percolation open for $d = 2$. For $d = 2$, 4-percolation is clear as there are no vertices of degree more than 4 and site percolation occurs on $\mathbb{Z}^2$ for large $p$. However this still leaves open the question of the existence of 4-percolation for small $d > 2$. 


Throughout this paper we shall use standard terminology from graph theory. In particular, \( V(G) \) and \( E(G) \) will denote the set of vertices, respectively edges, of the graph \( G \). If \( v \in V(G) \), \( N_G(v) \) will denote the set of neighbors of \( v \) in \( G \), and \( \deg_G(v) = |N_G(v)| \) will denote the degree of the vertex \( v \). Recall that a subgraph \( H \) of \( G \) is called induced if every edge of \( G \) whose endpoints lie in \( V(H) \) is also an edge of \( H \), and a set \( S \) of vertices is independent if there are no edges in \( G \) between vertices of \( S \).

2 \( k \)-percolation on trees

In this section we deal with the simple case of percolation on trees. Although the results of this section are relatively straightforward, they provide a number of instructive examples and set the stage for later sections.

For all \( k, d \geq 2 \), define

\[
 f_{k,d}(p) = \sum_{i=1}^{k-1} \binom{d-1}{i} ip^i(1-p)^{d-1-i}
\]

(1)

to be the expected value of a Binomial Bin\((d-1,p)\) random variable that has been truncated so that values \( \geq k \) are replaced with zero.

**Lemma 8.** Let \( T \) be a locally finite tree and let \( D = \{ \deg_T(v) : v \in V(T) \} \) be the set of degrees of vertices in \( T \). Fix \( 0 < p < 1 \). If \( f_{k,d}(p) > 1 \) for all \( d \in D \) then \( T_p^{\leq k} \) almost surely has an infinite component. If \( f_{k,d}(p) \leq 1 \) for all \( d \in D \) then \( T_p^{\leq k} \) almost surely does not have an infinite component.

**Proof.** Fix a root vertex \( v_0 \) of \( T \) and consider the subtree \( T' \) of \( T_p \) consisting of vertices \( v_n \) such that each of the vertices \( v_1, \ldots, v_{n-1} \) of the unique path \( v_0 \ldots v_n \) in \( T \) lies in \( T_p \) and has at most \( k \) neighbors in \( T_p \). Note that in particular we do not require \( v_n \) (or \( v_0 \)) to have at most \( k \) neighbors, so \( v_n \) may not lie in \( T_p^{\leq k} \). However, the event \( v_n \in V(T') \) depends only on the random process up to the level of \( v_n \). Fixing \( v_n \in V(T') \), \( n > 0 \), the number \( Y_{v_n} \) of children of \( v_n \) in \( T' \) is given by a Binomial distribution Bin\((\deg_T(v_n)-1,p)\), except that if this gives more than \( k \) vertices, then \( v_n \) has more than \( k \) neighbors in \( T_p \) (since \( v_{n-1} \) is also a neighbor) and so there are no children of \( v_n \) in \( T' \). Hence the mean number of children \( \mathbb{E}Y_{v_n} \) is given by \( \hat{f}_{k,\deg_T(v_n)}(p) \). Since the number of children of \( v_n \) is independent for different choices of \( v_n, T' \) is a ‘Galton-Watson’-like tree. The only slight complication is that the distribution of the number of children of \( v \in T' \) can vary depending on the degree of \( v \) in our original tree \( T \).

Let \( p_{n,v} \) be the probability of a path of length \( n \) existing in \( T' \) starting from \( v \) and going away from the root \( v_0 \). Then \( p_{n+1,v} = \mathbb{E}(1-\prod_i(1-p_{n,v_i})) \) where the product is over all
Thus from (1) we obtain

\begin{equation}
E(1 - (1 - \alpha)^{Y_v}) \geq \frac{1 - (1 - \alpha)^k}{k\alpha}(EY_v)\alpha.
\end{equation}

Now \(\frac{1 - (1 - \alpha)^k}{k\alpha} \to 1\) as \(\alpha \to 0\), so for sufficiently small \(\alpha\), \(\frac{1 - (1 - \alpha)^k}{k\alpha}(EY_v) \geq 1\) for every \(v\). Thus by induction on \(n\), \(p_{n,v} \geq \alpha\) for all \(n\) and \(v\). Now by compactness, there exists an infinite path in \(T'\) (and hence an infinite path in \(T^{\leq k}_p\)) with positive probability.

If \(f_{k,d}(p) \leq 1\) for all \(d \in D\), then \(f_{k,d}(p) = 1\) for only finitely many \(d\) and \(f_{k,d}(p) \leq 1 - \varepsilon\) for all the rest. A simple calculation shows that for \(\alpha \in (0, 1]\) there is a continuous function \(f(\alpha) < \alpha\) such that \(E(1 - (1 - \alpha)^{Y_v}) \leq f(\alpha)\) for every choice of distribution \(Y_v\). Indeed, \((1 - \alpha)^n \geq 1 - n\alpha + \alpha^2\) for all \(n \geq 2\), so \(E(1 - (1 - \alpha)^{Y_v}) \leq \alpha EY_v - \alpha^2 \mathbb{P}(Y_v \geq 2)\). Thus we can take \(f(\alpha) = \max\{(1 - \varepsilon)\alpha, \alpha - \alpha c^2\}\) where \(c\) is the minimum value of \(\mathbb{P}(Y_v \geq 2)\) over the finite set of distributions \(Y_v\) where \(EY_v = f_{k,d}(p) = 1\). Note that none of these distributions give 1 with certainty, so \(c > 0\). If we let \(p_n = \sup_v p_{n,v}\), then \(p_{n+1} \leq f(p_n)\), and so \(p_n \to 0\) as \(n \to \infty\). Hence \(T^{\leq k}_p\) almost surely has no infinite component.

Before we continue, we note a few properties of the functions \(f_{k,d}(p)\) (see Figure 1). An easy calculation shows that \(\frac{d}{dp} \binom{n}{i} p^i q^{n-i} = \frac{n}{p} \binom{n}{i} p^i q^{n-i} - \binom{n-1}{i} p^i q^{n-1-i}\), where \(q = 1 - p\). Thus from (1) we obtain

\begin{equation}
\frac{d}{dp} f_{k,d}(p) = \frac{d-1}{p}(f_{k,d}(p) - f_{k,d-1}(p)).
\end{equation}
For $d = k$, $f_{k,d}(p) = (d - 1)p$ is linear since it is just the expected value of a Binomial random variable $\text{Bin}(d-1, p)$. For $d > k$, $f_{k,d}$ is a unimodal function. Indeed, if $f_{k,d}$ had a local minimum at $p_0 \in (0, 1)$, say $f_{k,d}(p) = f_{k,d}(p_0) + (c + o(1))(p - p_0)^2n$, $c > 0$, then $f_{k,d-1}(p) = f_{k,d}(p_0) - (\frac{2np}{d-1} + o(1))(p - p_0)^{2n-1}$, so $f_{k,d-1}$ would be decreasing near $p_0$. But by induction on $d$ we may assume that $f_{k,d-1}$ is unimodal. Hence $f_{k,d-1}$ would be decreasing for all $p > p_0$, and so $f_{k,d}$ would be strictly increasing on $[p_0, 1]$, contradicting the fact that $f_{k,d}(1) = 0$. We also note that since the maximum value of $f_{k,d}$ occurs at a point where $f_{k,d} = f_{k,d-1}$, the maximum value of $f_{k,d}$ is a decreasing function of $d$.

**Proof of Theorem 3.** For $k = 3$, $f_{3,d}(p) = (d - 1)pq^{d-2} + (d - 1)(d - 2)p^2q^{d-3}$ where $q = 1 - p$. It is easy to show that $f_{3,d}(p) < 1$ for all $d \geq 6$ and all $p \in (0, 1)$. Indeed, by the above it is enough to prove this for the single function $f_{3,8}(p) = 5pq^4 + 20p^2q^3$ which has a maximum of $\frac{80}{81}$ at $p = \frac{1}{3}$. Hence Theorem 3 follows from Lemma 8. □

**Proof of Theorem 4.** Now $f_{3,3}(p) = 2p > 1$ for $p \in (0.5, 1)$, $f_{3,4}(p) = 3p(1 - p^2) > 1$ for $p \in [0.395, 0.742]$, and $f_{3,5}(p) = 4p(1 - 3p^2 + 2p^3) > 1$ for $p \in [0.345, 0.5]$ (see Figure 1). Lemma 8 then implies that the $d$-regular tree $T_d$ 3-percolates for $d = 3, 4,$ and 5. For $k = 4$ and large $d$ let $p = \frac{2}{d}$. Then $q^d = (1 - \frac{2}{d})^d \to e^{-2}$, so

$$f_{4,d}(p) = (d - 1)pq^{d-2} + (d - 1)(d - 2)p^2q^{d-3} + \frac{1}{2}(d - 1)(d - 2)(d - 3)p^3q^{d-4}$$

$$\quad \to (2 + 4 + 4)e^{-2} > 1 \quad \text{as } d \to \infty.$$ 

Since the maximum of $f_{4,d}$ is a decreasing function of $d$, we see that the maximum of $f_{4,d}$ must be greater than 1 for all $d \geq 3$. Hence by Lemma 8, $T_d$ 4-percolates for all $d \geq 3$.

For fixed $d \geq k \geq 4$, $f_{k,k}(p) > 1$ holds for all $p$ in some interval $(p_-, p_+)$ (or $(p_-, 1]$ if $d = k$). Both $p_+ = p_-(k, d)$ and $p_+ = p_+(k, d)$ are monotonically decreasing to 0 as a function of $d$. Hence if $f_{k,\delta}(p) > 1$ and $f_{k,\Delta}(p) > 1$ then $f_{k,\delta}(p) > 1$ for all $\delta \leq \Delta$. By Lemma 8, any tree with degrees between $\delta$ and $\Delta$ will $k$-percolate. Thus we can set $\Delta(k, \delta)$ to be the maximum $\Delta$ such that $p_+(k, \Delta) > p_-(k, \delta)$. For the converse, construct a tree level by level with branching ratio $\Delta = \Delta(k, \delta)$ on the levels $(2t)!$ to $(2t+1)! - 1$ and ratio $\delta - 1$ on levels $(2t+1)!$ to $(2t+2)! - 1$. For any $p \in (0, 1)$, either

$$\alpha := f_{k,\delta}(p) \leq 1 \quad \text{or} \quad \beta := f_{k,\Delta}(p) \leq 1.$$ 

If $\min\{\alpha, \beta\} = \alpha < 1$ then the expected number of paths to level $(2t + 2)!$ is at most $\alpha^{(2t+2)!}(\beta/\alpha)^{(2t+1)!} = (\alpha^{2t+1}\beta)^{(2t+1)!}$ which tends to 0 as $t \to \infty$. Similarly, if $\min\{\alpha, \beta\} = \beta < 1$ then the expected number of paths to level $(2t+1)!$ is at most $\beta^{(2t+1)!}(\alpha/\beta)^{(2t)!} = (\beta^{2t}\alpha)^{(2t)!}$ which tends to 0 as $t \to \infty$. In either case, the probability of a path of length $n$ from the root vertex tends to 0 as $n \to \infty$, so there is no percolation. If $\alpha = 1 < \beta$ or $\alpha > 1 = \beta$, then by continuity there exists a $p$ with $\alpha, \beta > 1$, so $p_+(k, \Delta + 1) > p_-(k, \delta)$, contradicting the choice of $\Delta$. Hence the only remaining case is when $\alpha = \beta = 1$, in which case the tree does not $k$-percolate by Lemma 8. □
Table 1: Values of $\Delta(k, \delta)$ for small $k$ and $\delta$.

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Table 2: Values of $\lambda_\pm$ for small $k$ and asymptotic values for large $k$.

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$\rightarrow \infty \quad 1 + \frac{e^{-\frac{\pm o(1)}{(k-1)!}}}{(k-1)!} \quad k + \sqrt{(2 + o(1)) k \log k}$

Table 1 lists $\Delta(k, \delta)$ for small $k$ and $\delta$. One can show that $\delta p_\pm \rightarrow \lambda_\pm$ as $\delta \rightarrow \infty$, where $\lambda_\pm$ are the two roots of the equation

$$\sum_{i=1}^{k-1} e^{-\lambda} \frac{\lambda^i}{(i-1)!} = 1.$$

Hence the interesting range of $p$ is when $p$ is of order $\frac{1}{d}$. The constants $\lambda_\pm$ for small $k$ and asymptotic values are given in Table 2. Since for large $d$, $p_\pm \sim \lambda_\pm / d$, $\Delta(k, \delta)/\delta \rightarrow \lambda_+/\lambda_-$ as $\delta \rightarrow \infty$. 
3 3-percolating graphs with large minimum degree

For trees, $\delta \geq 6$ implies that $G$ does not 3-percolate since for any $p \in (0, 1)$, $f_{3,d}(p) < 1$ when $d \geq 6$. However, it is possible for an arbitrary graph with large minimum degree to 3-percolate.

**Proof of Theorem 2.** We may assume $\delta > 3$ since otherwise we can just add a vertex of degree $\delta$ to the 3-regular infinite tree $T_3$. Fix $p = \frac{3}{4}$ and let $R$ be large. Construct $G$ as follows. Take $T_3$ and partition the vertices into subtrees $T_i$ isomorphic to the first $R$ levels of a binary tree (with $2^R - 1$ vertices). For example, take any edge $v_0v'_0$ of $T_3$ and consider $T_3$ as two back-to-back infinite binary trees $T$ and $T'$ rooted at $v_0$ and $v'_0$ respectively. In each of these binary trees take any vertex $v$ that is at distance a multiple of $R$ from the root and consider it as the root of a subtree $T_i$ consisting of $v$ and all its descendants for the next $R - 1$ levels. Then both $T$ and $T'$ are decomposed into trees $T_i$ as required. For each $i$ add $\delta - 3$ new vertices, and join them to every vertex of $T_i$ (see Figure 2). The minimum degree is now $\delta$ and the maximum degree is $\Delta = 2^R - 1$ (provided of course that $2^R - 1 \geq \delta$). By amalgamating each $T_i$ into a single vertex, $T$ and $T'$ can be thought of as trees of trees with “vertices” $T_i$, each of which has $2^R$ children. We count the number of paths from the root of $T_i$ to the roots of its child trees $T_j$ on the assumption that the $\delta - 3$ additional vertices do not lie in $G_p$. The total number of paths is $2^R$ and each is open (except possibly for the last vertex) with probability $p^R$. Thus the mean number of open paths is $p^{R}2^R = 1.5^R$. The mean number of paths across each $T_i$ to its children is then at least $0.25^{R-3}1.5^R$ which is $> 1$ if $R \geq 4\delta$. By comparison with a Galton-Watson process, regarding the $T_i$ as nodes, we almost surely have an infinite component.

Note that our example has a maximum degree of about $16^\delta$ which, as mentioned in the
introduction, is exponential in $\delta$.

4 Absence of 3-percolation

To prove Theorem 1 we shall use a path-counting technique, showing that the expected number of induced paths of length $n$ tends to zero as $n \to \infty$. The main complication is that one can encounter situations where, conditioned on the path getting to a certain vertex $v$, the number of possible extensions of this path to one new vertex is less than $\delta - 1$. This is due to the fact that, unlike for trees, many of the neighbors of $v$ may be adjacent to earlier vertices on the path, and so are ineligible for use in extending the path. Moreover, conditioned on the path existing in $G^{\leq k}_p$, these neighbors are very likely to be closed, and so do count to the degree of $v$ in $G_p$. Thus we may be in a situation that locally looks like a $k$-regular tree for $k \in \{3, 4, 5\}$, and we know that these do 3-percolate. However these situations cannot be too frequent as then some previous vertex in the path would have too many neighbors. To deal with these issues we shall introduce certain weighted counts of paths, and prove bounds inductively on these.

Proof of Theorem 1. Fix some vertex $v_0$ and, for $n > 0$, let $\mathcal{P}_n$ be the set of all induced paths $P = (v_0, v_1, \ldots, v_n)$ of $G$ of edge length $n$ starting at $v_0$ (so the only vertices of $P$ that are adjacent to $v_i$ in $G$ are $v_{i+1}$). Let $\mathcal{P}_n'$ be the (random) set of paths $(v_0, \ldots, v_n)$ of $\mathcal{P}_n$ that are subgraphs of $G_p$ and for which the degree in $G_p$ of $v_1, \ldots, v_{n-1}$ are all at most 3. Clearly, if the probability of an infinite component existing in $G^{\leq 3}_p$ is positive, then for some $v_0$ there must be a positive probability that there is an infinite component containing $v_0$. Hence there is an infinite induced path in $G^{\leq 3}_p$ starting at $v_0$ with probability at least some $\varepsilon_0 > 0$. Thus, in particular, $\mathbb{E} |\mathcal{P}_n'| \geq \varepsilon_0$ for all $n > 0$. We shall show that in fact $\mathbb{E} |\mathcal{P}_n'| < \varepsilon_0$ for sufficiently large $n$, so $G$ does not 3-percolate.

Fix a path $P = (v_0, \ldots, v_n) \in \mathcal{P}_n$. For $i = 0, \ldots, n-1$, let $d_i = d_i(P)$ be the number of vertices adjacent to $v_i$ that are not adjacent to $v_0, \ldots, v_{i-1}$. Thus $d_i$ is the number of possible choices for $v_{i+1}$ given $v_0, \ldots, v_i$. First we claim that

$$\sum_{P \in \mathcal{P}_n} \prod_{i=0}^{n-1} d_i^{-1} = 1. \quad (3)$$

Indeed, there are $d_{n-1}$ paths $(v_0, \ldots, v_n) \in \mathcal{P}_n$ for each path $(v_0, \ldots, v_{n-1}) \in \mathcal{P}_{n-1}$, so $\sum_{P \in \mathcal{P}_n} \prod_{i=0}^{n-1} d_i^{-1} = \sum_{P' \in \mathcal{P}_{n-1}} d_{n-1} \prod_{i=0}^{n-2} d_i^{-1} = \sum_{P' \in \mathcal{P}_{n-1}} \prod_{i=0}^{n-2} d_i^{-1} = 1$ by induction on $n$. The case $n = 1$ of (3) is of course trivial.

Consider the event that $P \in \mathcal{P}_n'$. Let $N_n$ be the set of neighbors of $\{v_1, \ldots, v_{n-1}\}$ in $G$ that do not lie in $P$. Then $P \in \mathcal{P}_n'$ iff $V(P) \subseteq V(G_p)$ and for all $i = 1, \ldots, n-1$ there
Figure 3: Path $P$ and auxiliary graph $H_n$ on vertex set $N_n$. The vertex $v_{n-1}$ has $s = 1$ ‘old’ neighbors in $N_{n-1}$ and $r = 2$ ‘new’ neighbors in $N' = N_n \setminus N_{n-1}$. Edges exist in $H_n$ between vertices with common neighbors in $\{v_1, \ldots, v_{n-1}\}$.

is at most one vertex of $N_G(v_i) \cap N_n$ lying in $G_p$. (Since $P$ is induced, each of these $v_i$ is adjacent to precisely two vertices of $P$.) Construct an auxiliary graph $H_n$ with vertex set $N_n$ and edges joining two vertices $u$ and $v$ if and only if $u$ and $v$ have a common neighbor in $\{v_1, \ldots, v_{n-1}\}$ (see Figure 3). Thus for any $P \in \mathcal{P}_n$ we have $P \in \mathcal{P}'_n$ if and only if $V(P) \subseteq V(G_p)$ and $V(G_p) \cap N_n$ is an independent set in the graph $H_n$. Hence

$$
\mathbb{P}(P \in \mathcal{P}'_n) = p^{n+1} \sum_{I \text{ independent}} \mathbb{P}(V(G_p) \cap N_n = I) = p^{n+1} \sum_{I} \prod_{v \in N_n} z(I, v),
$$

where

$$
z(I, v) = \begin{cases} 
1 - p, & \text{if } v \notin I; \\
p, & \text{if } v \in I;
\end{cases}
$$

and the sum runs over all independent sets $I$ of $H_n$.

We aim to bound this expression by induction on $n$, but to do this we shall need to consider a slight generalization. Fix a weight function $w: N_n \to \mathbb{Z}$ such that for all $v \in N_n$, $0 \leq w(v) \leq \Delta - 1$, and fix constants $\alpha \geq 1$ and $\beta \leq \alpha$, (to be determined). Define the partition function

$$Z_{n,w} = \sum_{I} \prod_{v \in N_n} z_w(I, v),
$$

where

$$z_w(I, v) = \begin{cases} 
q \alpha^w(v), & \text{if } v \notin I; \\
p \beta^w(v), & \text{if } v \in I;
\end{cases}
$$

and $q = 1 - p$. 

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Let \( r = d_{n-1} - 1 \) and \( s = \deg_G(v_{n-1}) - 2 - r \). Then \( |N_G(v_{n-1}) \cap N_n| = r + s \) with \( r = |N_n \setminus N_{n-1}| \) ‘new’ neighbors and \( s = |N_G(v_{n-1}) \cap N_{n-1}| \) ‘old’ neighbors. Define \( w' : N_{n-1} \to \mathbb{Z} \) by

\[
w'(v) = \begin{cases} 
  w(v) + 1, & \text{if } v \in N_G(v_{n-1}); \\
  w(v), & \text{if } v \notin N_G(v_{n-1}).
\end{cases}
\]

We shall compare \( Z_{n,w} \) with \( Z_{n-1,w'} \). Fix an independent set \( I_{n-1} \) of \( H_{n-1} \). There are two cases. Assume first that \( I_{n-1} \cap N_G(v_{n-1}) = \emptyset \). Then any extension of \( I_{n-1} \) to an independent set \( I_n \) of \( N_n \) can contain at most one vertex from \( N' := N_n \setminus N_{n-1} \). Thus

\[
\sum_{I_n : I_n \cap N_{n-1} = I_{n-1}} \prod_{v \in N_n} z_w(I_n, v) \leq \prod_{v \in N_n} z_w(I_{n-1}, v) + \sum_{u \in N'} \prod_{v \in N_n} z_w(I_{n-1} \cup \{u\}, v)
\]

\[
= \left( \prod_{v \in N'} q\alpha^{w(v)} \right) \left( 1 + \sum_{u \in N'} \frac{p\beta^{w(u)}}{q\alpha^{w(u)}} \right) \prod_{v \in N_{n-1}} z_w(I_{n-1}, v)
\]

\[
= q^r \alpha^{\sum_{v \in N'} w(v)} \left( 1 + \sum_{u \in N'} \frac{p\beta^{w(u)}}{q\alpha^{w(u)}} \right) \alpha^{-s} \prod_{v \in N_{n-1}} z_w'(I_{n-1}, v).
\]

(This is an inequality as \( I_{n-1} \) may not be independent in \( H_n \) and so the sum on the left hand side may be empty. In the last line we are using the fact \( w'(v) = w(v) + 1 \) only when \( v \in N_G(v_{n-1}) \), but then \( v \notin I_{n-1} \).) On the other hand, if \( I_{n-1} \cap N_G(v_{n-1}) \neq \emptyset \) then the only possible extension is \( I_n = I_{n-1} \). Also \( |I_{n-1} \cap N_G(v_{n-1})| = 1 \), so

\[
\prod_{v \in N_n} z_w(I_n, v) = q^r \alpha^{\sum_{v \in N'} w(v)} \alpha^{1-s} \beta^{-1} \prod_{v \in N_{n-1}} z_w'(I_{n-1}, v).
\]

Now assume that

\[
p(r + 1)q^r \alpha^{\sum_{v \in N'} w(v)} \left( 1 + \sum_{v \in N'} \left( \frac{p\beta^{w(v)}}{q\alpha^{w(v)}} \right) \right) \leq (1 - \varepsilon)\alpha^s \tag{5}
\]

and

\[
p(r + 1)q^r \alpha^{\sum_{v \in N'} w(v)} \leq (1 - \varepsilon)\alpha^{s-1} \tag{6}
\]

Then in both cases above,

\[
p(r + 1) \sum_{I_n : I_n \cap N_{n-1} = I_{n-1}} \prod_{v \in N_n} z_w(I_n, v) \leq (1 - \varepsilon) \prod_{v \in N_{n-1}} z_w(I_{n-1}, v)
\]

so that

\[
pd_{n-1}Z_{n,w} = p(r + 1) \sum_{I_n} \prod_{v \in N_n} z_w(I_n, v)
\]

\[
\leq (1 - \varepsilon) \sum_{I_{n-1}} \prod_{v \in N_{n-1}} z_w'(I_{n-1}, v) = (1 - \varepsilon)Z_{n-1,w'}.
\]
Now fix \( n \) and \( w = w_n = 0 \). By induction
\[
p^{n-1} \left( \prod_{i=1}^{n-1} d_i \right) Z_n \leq (1 - \varepsilon)^{n-1} Z_{1,w_1},
\]
where each \( w_i = (w_{i+1})' \) used in the induction satisfies \( 0 \leq w_i(v) \leq \deg_G(v) - 1 \leq \Delta - 1 \). Indeed, \( w_i(v) \) is just the number of neighbors of \( v \) in \( \{v, v_{i+1}, \ldots, v_{n-1}\} \). But \( N_1 = \emptyset \) and so \( Z_{1,w_1} = 1 \). Thus
\[
E|\mathcal{P}'_n| = \sum_{P \in \mathcal{P}'_n} \mathbb{P}(P \in \mathcal{P}'_n) = p^{n+1} \sum_{P \in \mathcal{P}_n} Z_n
\leq p^2 d_0 (1 - \varepsilon)^{n-1} \sum_{P \in \mathcal{P}_n} \left( \prod_{i=0}^{n-1} d_i^{-1} \right) Z_{1,w_1}
\leq p^2 \Delta (1 - \varepsilon)^{n-1},
\]
where in the last step we have used (3). Thus for sufficiently large \( n \), \( E|\mathcal{P}'_n| < \varepsilon_0 \) contradicting our assumption that \( E|\mathcal{P}'_n| \geq \varepsilon_0 \).

It remains to find \( \alpha \geq 1 \) and \( \beta \leq \alpha \) satisfying (5) and (6) for all weight functions \( w \) with \( 0 \leq w(v) \leq \Delta - 1 \). Let
\[
A = \alpha^{\Delta-1}, \quad B = \beta^{\Delta-1}.
\]
By convexity of \( \alpha^i \) and \( \beta^j \), (5) and (6) hold if they hold for all choices of weight function with \( w(v) \in \{0, \Delta - 1\} \). Setting \( i \) to be the number of \( v \in N' \) with \( w(v) = \Delta - 1 \), it is enough if for all \( 0 \leq i \leq r, s \geq 0 \), and \( \delta - 2 \leq r + s \leq \Delta - 2 \),
\[
p(r+1)q^{r-1} A^i (q + (r - i)p + ipB/A) \leq (1 - \varepsilon) \alpha^s
\]
\[
p(r+1)q^r A^i \leq (1 - \varepsilon) \alpha^{s+1} \beta.
\]
Since there are only a finite number of inequalities, it is enough if
\[
p(r+1)q^{r-1} A^i (q + (r - i)p + ipB/A) < \alpha^{\max(\delta-r-2,0)}
\]
\[
p(r+1)q^r A^i < \alpha^{\max(\delta-r-2,0) - 1} \beta.
\]
where we have used \( \alpha \geq 1 \) and substituted the smallest possible value of \( s \) and, in (12), the largest possible value of \( i \). The result now follows from the following lemma. \( \square \)

**Lemma 9.** For all \( 6 \leq \delta \leq \Delta \leq 2.372(\delta - 3), (\delta, \Delta) \neq (6,7) \), and for all \( p \in (0,1) \), there exist \( \alpha \geq 1 \) and \( \beta \leq \alpha \) satisfying (11) and (12) for all \( 0 \leq i \leq r \leq \Delta - 2 \), where \( A \) and \( B \) are defined as in (8).
Proof. First consider (11) with $i = 0$.

$$f_{3,r+2} = p(r + 1)q^{r-1}(q + rp) < \alpha^\max\{\delta - r - 2, 0\}, \quad (13)$$

where $f_{k,d}$ is as in Section 2. For $r \not\in \{1, 2, 3\}$ the left hand side is $< 1$ for all $p$. If the left hand side is also $< 1$ for $r \in \{1, 2, 3\}$ then we can take $\alpha = \beta = 1$ and then all the inequalities (11) and (12) hold. Now assume the left hand side is $\geq 1$ for some $r \in \{1, 2, 3\}$. We shall make the following assumptions.

$$\alpha(qA + pB) \leq \frac{2}{3}, \quad \beta = \frac{2}{3}. \quad (14)$$

We shall show that all the inequalities (11) and (12) follow from (13) and (14). For $r \geq 2$, $i \geq 1$, inequality (11) with parameters $(r, i)$ follows from the $(r - 1, i - 1)$ case of (11). Indeed,

$$p(r + 1)q^{r-1}A^i(q + (r - i)p + iP/B/A))$$

$$\leq p(r + 1)q^{r-1}A^i(q + (r - i)p + (i - 1)pB/A)(1 + pB/qA)$$

$$\leq \frac{r+1}{r}qA\alpha^\max\{\delta - r - 1, 0\}(1 + pB/qA) \quad \text{from } (r - 1, i - 1) \text{ case}$$

$$\leq \frac{2}{3}(qA + pB)\alpha^\max\{\delta - r - 1, 0\} \quad \text{since } r \geq 2$$

$$\leq \alpha^\max\{\delta - r - 2, 0\} \quad \text{by } (14) \text{ and } \alpha \geq 1$$

For $r \geq 2$ the inequalities (12) follow the $r - 1$ case of (12):

$$p(r + 1)q^{r-1}A^r \leq \left(\frac{2}{3}qA\right)(prq^{r-1}A^{r-1}) < \alpha^{-1}\alpha^\max\{\delta - r - 1, 0\}^{-1}\beta \leq \alpha^\max\{\delta - r - 2, 0\}^{-1}\beta.$$}

The $i = 0$ case of (11) is of course precisely (13), and the remaining inequalities follow from the $r = 1$ case of (13), i.e., $2p < \alpha^{\delta - 3}$ (note $\delta \geq 6$). For the $(r, i) = (1, 1)$ case of (11)

$$2pA(q + pB/A)) \leq 2p(qA + pB) \leq \frac{2}{3p}2p < 2p < \alpha^{\delta - 3},$$

and for the $r = 1$ and $r = 0$ cases of (12)

$$2pqA \leq \left(\frac{2}{3p}\right)2p < \frac{2}{\alpha}\alpha^{\delta - 3} = \alpha^{\delta - 4}\beta, \quad \text{and} \quad p < \left(\frac{3}{2}\right)2p < \alpha^{\delta - 3}\beta.$$}

Hence we are done provided (13) and (14) both hold. By setting $\alpha$ to be slightly larger than the maximum of $f_{3,k}^{1/(\delta - k)}$, $k = r + 2 \in \{3, 4, 5\}$, (so that (13) holds) it is then enough if

$$qf_{3,k}^{\Delta/(\delta - k)} < \frac{2}{3} - 2\left(\frac{2}{3}\right)^{\Delta - 1} \quad (15)$$

whenever $f_{3,k} \geq 1$ (using the trivial bound $\alpha p \leq \alpha \leq f_{3,k} \leq 2$).

For $\Delta \geq \delta \geq 60$, say, (15) holds when $qf_{3,k}^{\Delta/(\delta - k)} < 0.6666$. These inequalities impose a restriction on $\Delta$ only via the ratio $\Delta/(\delta - k)$, and by maximizing the left hand side over $p \in (0, 1)$ it is enough that

$$\Delta \leq \min\{2.372(\delta - 3), 2.415(\delta - 4), 2.569(\delta - 5)\} = 2.372(\delta - 3).$$
For $\delta < 60$, inequalities (11) and (12) were checked directly by computer, choosing $\alpha$ and $\beta$ as above. To deal with the fact that there are infinitely many values of $p$, we divided $[0, 1]$ into small ranges $[p_k, p_{k+1}]$ and checked the inequalities (11) and (12) for all $p$ in this range by setting $p = p_{k+1}$ and $q = 1 - p_k$ so as to bound the left hand sides of (11) and (12) for all $p \in [p_k, p_{k+1}]$. All cases for which $\delta \leq \Delta \leq 2.372(\delta - 3)$ and $60 > \delta \geq 6$ were successful, except for the case $(\delta, \Delta) = (6, 7)$, which fails for, e.g., $p = 0.5$. \qed

We finish this section with a useful Lemma.

**Lemma 10.** For sufficiently large $\delta$ and $\Delta/\delta$ sufficiently close to 1, the probability of an induced path of $G_p^{< \infty}$ of edge length $n$ starting at any given point of $G$ is at most $p^2 \Delta(0.842)^{n-1}$. 

**Proof.** Fix a point $v_0$ of $G$. Then in the proof of Theorem 1 we have by (7)

$$\mathbb{P}(\exists \text{ an induced path of length } n \text{ starting at } v_0 \text{ in } G_p^{\leq k}) \leq \mathbb{E}[\mathcal{P}_n^k] \leq p^2 \Delta(0.842)^{n-1},$$

provided that we can take $1 - \epsilon = 0.842$ in (9) and (10). Thus it is enough to find $\alpha \geq 1$ and $\beta \leq \alpha$, depending only on $p$, such that for all $0 \leq i \leq r$,

$$p(r + 1)q^{r-1}A^i(q + (r - i)p + ipB/A) \leq ca^{\max\{\delta - r - 2, 0\}},$$

$$p(r + 1)q^rA^r \leq ca^{\max\{\delta - r - 2, 0\} - 1}\beta,$$  

where $c = 0.842$. Comparing (17) with the case $i = r$ of (16), we see that if we reduce $c$ slightly to $c_1 < c$ in (16) we can take $\beta = (c_1/c)\alpha$ so that (17) follows. Then $B/A = (c_1/c)^{\Delta^{-1}}$ tends to 0 as $\Delta \to \infty$. Thus for any fixed $\gamma < 1$ it is enough that

$$p(r + 1)q^{r-1}A^i(q + (r - \gamma i)p) \leq c_1a^{\max\{\delta - r - 2, 0\}}.$$  

First we deal with the case when $p$ is very small. In this case take $\alpha = 1$. The left hand side of (18) is maximized when $i = 0$, so we are reduced to checking the inequality $p(r+1)q^{r-1}(q+rp) \leq c_1$. Let $x = (r-1)p$. Then $p(r+1)q^{r-1}(q+rp) \leq (2p+x)e^{-x}(1+x) \leq 2p + x(x+1)e^{-x}$. This last expression is maximized when $x = (\sqrt{5} + 1)/2$ and is less than $2p + 0.840$. Since $c_1$ can be chosen arbitrarily close to 0.842, the result follows for all $p < 0.001$. For other $p$ we shall choose $\alpha$ so that $1 \leq A = a^{\Delta^{-1}} \leq 1 + p + p^2$, so $qA$ is bounded away from 1. Then $p(r + 1)q^{r-1}A^i(q + (r - \gamma i)p) \leq 3(r + 1)^2(qA)^{r-1}$ tends rapidly to 0 as $r$ grows. Thus (18) automatically holds for all large $r$. Since $r$ and $A$ are bounded and $\delta/\Delta$ is close to 1, the right hand side is close to $c_1 A$. Thus we are reduced to finding $A$ with

$$p(r + 1)q^{r-1}A^i(q + (r - \gamma i)p) \leq c_2 A,$$  

where we have reduced $c$ a bit further to $c_2 < c_1$. This inequality is independent of $A$ when $i = 1$, and it is this case that puts the most stringent bounds on $c_2$ (and hence $c$).
Maximizing \( p(r + 1)q^{r-1}(q + (r - 0.99999)p) \) over all \( r \) and \( p \) gives a value just under 0.8419 with \( r = 16 \). Thus we take \( c_2 = 0.8419 \). To satisfy (19) when \( i = 0 \) we set \( A = c_2^{-1} \sup_r p(r+1)q^{r-1}(q+rp) \). (If this is less than 1 then we can take \( A = 1 \) and then (19) holds for all \( i \).) It now remains to check that \( A \leq 1 + p + p^2 \) and (19) also holds for all \( i \geq 2 \). We checked this by computer. To deal with the fact that there are infinitely many values of \( p \), we once again divided \([0,1]\) into small ranges \([p_k,p_{k+1}]\) and checked (19) for all \( p \) in this range by setting \( p = p_{k+1} \) and \( q = 1 - p_k \), so as to bound the left hand side of (19) for all \( p \in [p_k,p_{k+1}] \). The value of \( A \) is chosen to be that for \( p = p_{k+1} \), since it can be shown that \( A \) is an increasing function of \( p \). Only a finite number of pairs \((r,i)\) need to be checked as (19) automatically holds for large \( r \) since \( qA < 1 \).

\[ \square \]

## 5 Continuum Percolation

**Proof of Theorem 5.** Let the vertices of the graph \( G = G_A(\lambda) \) be given by a uniform Poisson process in \( \mathbb{R}^d \) of intensity \( \lambda \), and let \( uv \) be an edge of \( G \) whenever \( u - v \) lies in a fixed symmetric bounded set \( A \subseteq \mathbb{R}^d \) of positive measure \( |A| \). As selecting points independently with probability \( p \) from a Poisson process of intensity \( \lambda/p \) yields a Poisson process of intensity \( \lambda \), we can regard \( G \) as \( G'_p \) where \( G' = G_A(\lambda/p) \). Let \( R \) be a bound on the length of an edge, so for example we can take \( R = \sup_{x \in A} ||x|| \). Pick \( \varepsilon > 0 \). Let \( t > 0 \) be chosen so that \( 0.842^t < \varepsilon \). The number of neighbors \( N \) of a vertex of \( G' \) is given by a Poisson distribution with mean and variance \( \mu = |A|\lambda/p \). Now by the Chernoff bound

\[ \mathbb{P}(|N - \mu| \geq \varepsilon\mu) \leq 2e^{-\varepsilon^2\mu/3} \]

for all \( \varepsilon \leq 1 \). Thus for fixed \( \varepsilon \leq 1 \), the probability that the degree of a vertex of \( G' \) is outside the range from \((1 - \varepsilon)|A|\lambda/p\) to \((1 + \varepsilon)|A|\lambda/p\) is bounded by \( 2e^{-\varepsilon^2|A|\lambda/(3p)} \).

The expected number of vertices within \( tR + 1 \) of the origin is at most \((2tR + 1))^d\lambda/p \). Moreover, as the vertices are distributed according to a Poisson process, conditioning on the presence of a vertex at a point does not affect its degree distribution. Thus by linearity of expectation, the expected number of vertices within \( tR + 1 \) of the origin with exceptional degrees is bounded by \( 2e^{-\varepsilon^2|A|\lambda/(3p)}(2(tR + 1))^d\lambda/p \) which tends rapidly to zero as \( p \to 0 \). Thus we can choose \( p > 0 \) sufficiently small so that with probability at least \( 1 - \varepsilon \), all points of \( G' \) within \( tR + 1 \) of the origin have degree between \((1 - \varepsilon)|A|\lambda/p \) and \( \Delta := (1 + \varepsilon)|A|\lambda/p \). Let \( B \) be the unit ball about the origin, and note that there are on average \(|B|\lambda/p\) vertices in \( B \). By Lemma 10 applied to the part of the graph \( G' \) within distance \( tR + 1 \) of the origin, and assuming \( \varepsilon \) is sufficiently small, the probability of a path of length \( t \) originating from some point in \( B \) is at most \(|B|\lambda/p)^2\Delta 0.842^{t-1} = (1 + \varepsilon)|A||B|\lambda^2 0.842^{t-1} \). Once again we have used linearity of expectation and the fact that conditioning on a vertex being present at a point does not affect the distribution.
of the remaining points in a Poisson process. Thus the probability of a path of length $t$ starting in $B$ is at most $2|A||B|\lambda^2 0.842^{t-1} + \varepsilon = O(\varepsilon)$. The probability of an infinite component intersecting $B$ is therefore $O(\varepsilon)$. Since $\varepsilon$ is arbitrary, there is almost surely no infinite component.

\begin{proof}
Assume the Poisson process has intensity $\lambda$ and $R > R_0/\sqrt{\lambda}$ where $R_0 = 2.59$. We fix $p = (R_0/R)^2\lambda^{-1}$, so that $G_p$ is given by a Poisson process with intensity $(R_0/R)^2$. By scaling $\mathbb{R}^2$ we may assume without loss of generality that $G_p$ has intensity 1 and that $R = R_0$. Divide $\mathbb{R}^2$ into $3R \times 3R$ squares, and identify each square with an element of $\mathbb{Z}^2$ in the obvious way. We shall define a bond percolation process on $\mathbb{Z}^2$ so that percolation in this process implies percolation in $G_p\leq 36$. The bonds in this percolation will not be independent, but will be $1$-independent — any two sets $S_1$ and $S_2$ of bonds that are at graph distance at least 1 in $\mathbb{Z}^2$ from each other are independent. By Theorem 2 of [1], if the probability of each bond being open in a $1$-independent process on $\mathbb{Z}^2$ is at least 0.8639, then with positive probability there is an infinite open path from the origin in $\mathbb{Z}^2$.

Consider two neighboring squares corresponding to an edge $xy$ of $\mathbb{Z}^2$. Let $A$ and $C$ be disks of diameter $R$ in the center of the squares corresponding to $x$ and $y$ respectively. Let $B$ be the set of points lying directly between $A$ and $C$ (see Figure 4). Let $E_{xy}$ be the event that every vertex of $G_p$ in the region $A \cup B \cup C$ has at most 36 neighbors, there exists at least one vertex of $G_p$ in $A$, and every vertex $v \in A \cup B$ has a neighbor in $D_v$, where $D_v$ is the disk of diameter $R$ in $A \cup B \cup C$ with $v$ on its leftmost boundary. The expected number of $v \in A \cup B \cup C$ with more than 36 neighbors in $G_p$ is

$$\left| A \cup B \cup C \right| \mathbb{P}(\text{Pois}(\pi R^2) > 36) = (3 + \frac{\pi}{4})R^2 \mathbb{P}(\text{Pois}(\pi R^2) > 36)$$

where Pois($\mu$) is a Poisson variable with mean $\mu$. The probability that there is no vertex of $G_p$ in $A$ is $e^{-\pi R^2/4}$. The expected number of $v \in A \cup B$ with no vertex in $D_v$ is $|A \cup B| e^{-\pi R^2/4} = 3R^2 e^{-\pi R^2/4}$. Thus (for $R = 2.59$)

$$\mathbb{P}(E_{xy} \text{ fails}) \leq (3 + \frac{\pi}{4})R^2 \mathbb{P}(\text{Pois}(\pi R^2) > 36) + (1 + 3R^2) e^{-\pi R^2/4} < 0.1357.$$
Now define the bond percolation process in $\mathbb{Z}^2$ by declaring $xy$ open if $E_{xy}$ holds. If $E_{xy}$ holds then every vertex in $A$ is joined by a path in $G_{36}^{\leq}$ to one (and hence every) point in $C$, and there is at least one vertex in each of $A$ and $C$. An infinite open path in $\mathbb{Z}^2$ therefore gives rise to an infinite path in $G_{36}^{\leq}$. The bond percolation on $\mathbb{Z}^2$ is $1$-independent as the event $E_{xy}$ depends only on the vertices of $G_p$ inside the union of the squares corresponding to $x$ and $y$. Since $1 - 0.1357 > 0.8639$, Theorem 2 of [1] implies that $G_{36}^{\leq}$ almost surely has an infinite component.

6 Percolation on a lattice

In this section we shall prove Theorem 7. We shall follow the method used by Cox and Durrett [2] to prove upper bounds for oriented percolation in high dimensions. We shall need to modify this approach somewhat, since we are dealing with unoriented percolation as well as interference. The basic idea is show that there are long paths by bounding below the expected number of, and above the variance of, the number of paths of length $k$. Then with reasonable probability a path exists. To deal with the case when the paths are unoriented, we shall apply this method twice. The first time we shall show that with reasonable probability there are a large number of ‘short’ unoriented paths in $d_0 = d - 8$ dimensions. To avoid combinatorial problems associated with these paths looping back on themselves, we shall insist that they at move at most one step in any dimension (although they can go either direction in each dimension). Since we will quickly run out of dimensions, the length $k$ of these paths will be very small compared with $d$. We then show that with reasonable probability we can extend some of these paths by three steps in each of the remaining 8 dimensions. Having done this, we repeat the process starting at these 8 new locations. Continuing in this manner, we show that we can couple the process with an oriented independent site percolation in $\mathbb{Z}^8$. We then use the arguments of [2] a second time to bound the critical probability of this process.

Let $\{e_1, \ldots, e_d\}$ be the standard basis of unit vectors for $\mathbb{Z}^d$. Fix integers $d_0 = d - 8$ and $k$ and let $\mathcal{P}$ be the set of paths $P = (v_0, v_1, \ldots, v_k)$ where $v_0 = 0 \in \mathbb{Z}^d$, $v_i = v_{i-1} \pm e_{d_i}$ and the $d_i$ are distinct. In other words, $P$ is a path in $\{-1,0,1\}^{d_0} \times \{0\}^8$ from the origin such that the $\ell_1$-distance from $v_0$ to $v_i$ is exactly $i$. Fix $p$ and let each vertex of $\mathbb{Z}^d$ be open independently with probability $p$.

Lemma 11. Conditioned on the event that $P_0 = (v_0, \ldots, v_k) \in \mathcal{P}$ is open, the probability that $P_0$ is the unique open path of $\mathcal{P}$ from $v_0$ to $v_k$ is at least $1 - 4kp$.

Proof. We may assume without loss of generality that $v_i = e_1 + e_2 + \cdots + e_i$ for $0 \leq i \leq k$, so that $v_i = v_{i-1} + e_i$. Any other path in $\mathcal{P}$ with endpoint $v_k$ must be of the form $v_i = v_{i-1} + e_{\pi(i)}$ for some permutation $\pi$ of the numbers $\{1, \ldots, k\}$. The conditional
probability that this path is open is just \( p^r \) where \( r \) is the number of vertices in the path specified by \( \pi \) that are not in \( P_0 \). We now count the number of permutations \( \pi \) which give \( r \) new vertices. The permutation \( \pi \) can be specified by specifying the choice of steps \( t \) at which these new vertices occur, and for each such \( t \) the value of \( \pi(t) \). The values of \( \pi(t) \) at all the other steps are then determined, since the vertex reached must be the same as for \( P_0 \). There are \( \binom{k-1}{r} \) choices for these steps. At the \( i \)th such step \( t \), \( \pi(t) \) must be chosen so that no \( s \) with \( s > t + (r - i + 1) \) occurs in the set \( \{ \pi(1), \ldots, \pi(t) \} \), otherwise the next \( r - i + 1 \) steps will also lead to new vertices, giving more than \( r \) new vertices in total. Thus \( \{ \pi(1), \ldots, \pi(t) \} \subseteq \{1, \ldots, t + (r - i + 1)\} \). Hence, given \( \{ \pi(1), \ldots, \pi(t - 1) \} \), there are at most \( r - i + 2 \) remaining choices for \( \pi(t) \) and so the total number of choices for \( \pi \) is at most \( \binom{k-1}{r} (r + 1)! \leq (r + 1)k^r \). Hence the expected number of such open paths (other than \( P_0 \) itself) is at most

\[
\sum_{r=1}^{\infty} (r + 1)k^r p^r = \frac{kp}{(1 - kp)^2} + \frac{kp}{1 - kp},
\]

which is at most \( (\frac{16}{9} + \frac{4}{3})kp < 4kp \) when \( 4kp < 1 \). Hence the probability that \( P_0 \) is not unique is at most \( 4kp \) when \( 4kp < 1 \), and this trivially also holds when \( 4kp \geq 1 \).

We shall set \( p = 1/d \). Since each vertex of \( \mathbb{Z}^d \) has \( 2d \) neighbors, this implies that each vertex has \( 2 \) open neighbors on average. Write \( q = 1 - p \) and define

\[
\alpha_{d,K} = q^{2d-K} + \binom{2d-K}{1}pq^{2d-K-1} + \binom{2d-K}{2}p^2q^{2d-K-2}
\]

to be the probability that at most \( 2 \) of some given set of \( 2d - K \) sites are open. Note that for any fixed \( K \)

\[
\alpha_{d,K} \to \alpha_{\infty} := e^{-2}(1 + 2 + 2^2/2) = 5e^{-2} > 0.6766 \quad \text{as } d \to \infty.
\]

Fix \( P = (v_0, \ldots, v_k) \in \mathcal{P} \) and define the following events.

\begin{align*}
\mathcal{E}_s: & \quad P \text{ is open;} \\
\mathcal{E}_u: & \quad \text{no path } P' \in \mathcal{P} \setminus \{ P \}, \text{ joining } v_0 \text{ to } v_k \text{ is open;} \\
\mathcal{F}_i: & \quad \text{the vertex } v_i \text{ has at most } 4 \text{ open neighbors in } \mathbb{Z}^d \ (0 \leq i < k); \\
\mathcal{G}: & \quad \text{there are no open vertices } v \notin \mathbb{Z}^{d_0} \times \{0\}^8 \text{ with } \|v - v_k\|_1 = 2 \text{ and } \|v - v_0\|_1 = k.
\end{align*}

Let \( I_P \) be the indicator function of the event that \( P \) satisfies all these conditions. Note that we don’t insist that \( v_k \) has few open neighbors. The set of points that we require to be closed for \( \mathcal{G} \) to hold has cardinality \( 16k \) as there are 16 directions from \( v_k \) out of the
hyperplane $\mathbb{Z}^{d_0} \times \{0\}^8$, but then we must move one step closer to $v_0 = 0$ and so must reverse one of the $k$ steps taken along the path $P$ from $v_0$ to $v_k$.

Let $S$ be a set of at most $d^6$ points in $\mathbb{Z}^{d_0} \times \{0\}^8$, all of which are at $\ell_1$-distance at least 6 from $v_0$. Let $X_S = \sum_{P \in \mathcal{P}, P \cap S = \emptyset} I_P$, so that $X_S$ is a lower bound on the number of vertices reached in $k$ steps from 0 by open paths avoiding the set $S$. Let $\mathcal{H}$ be the event that both vertices 0 and $-e_d$ are open in $\mathbb{Z}^d$, and that $-e_d$ has no open neighbors in $\mathbb{Z}^d$ other than 0 and $-2e_d$. The next lemma shows that our model grows rapidly in the first few steps.

**Lemma 12.** Fix $p = 1/d$ and $k = \lfloor 40 \log d \rfloor$. Then for sufficiently large $d$, $\mathbb{P}(X_S \geq d^{12} | \mathcal{H}) \geq 0.1766$.

**Proof.** We shall deduce the result from bounds on the first and second moments of $X_S$. We first find a lower bound for $\mathbb{E}(I_P | \mathcal{H})$. Now $\mathbb{P}(E_o | \mathcal{H}) = p^k$ and $\mathbb{P}(G | E_o \cap \mathcal{H}) = q^{16k}$ as $E_o$ requires $k$ additional vertices to be open and $G$ requires a disjoint set of 16 vertices to be closed, none of which are affected by $\mathcal{H}$. Also $\mathbb{P}(F_i | E_o \cap \mathcal{H}) \geq \alpha_{d,2}$ as all vertices $v_i$ have two open neighbors on path, so can have at most 2 more open neighbors out of $2d - 2$ remaining neighbors. (For $i = 1$ there is an inequality as $\mathcal{H}$ implies that one neighbor of $v_1$ is known to be closed.) Lemma 11 implies that $\mathbb{P}(E_u | E_o \cap \mathcal{H}) \geq 1 - 4kp$. Note that $E_u$ depends only on the status of vertices in $\mathbb{Z}^{d_0} \times \{0\}^8$, so conditioning on $\mathcal{H}$ is equivalent to conditioning on $v_0$ being open, which is included in the event $E_o$. Now conditioning on $E_o$ and $\mathcal{H}$ (which just fixes the states of a certain set of vertices), $E_u$, $F_i$, and $G$ are all monotone events, decreasing as we increase the set of open vertices. Thus by the FKG inequality,

$$\mathbb{P}(I_P = 1 | E_o \cap \mathcal{H}) = \mathbb{P}(E_u \cap \bigcap_i F_i \cap G | E_o \cap \mathcal{H}) \geq (1 - 4kp)\alpha_{d,2}^k q^{16k}.$$

Thus as $k = o(d)$ and $q = 1 - 1/d$,

$$\mathbb{E}(I_P | \mathcal{H}) \geq (1 - 4kp)\alpha_{d,2}^k q^{16k} p^k = (1 - o(1))(p\alpha_{d,2})^k.$$

Now

$$|P| = 2^k d_0(d_0 - 1) \ldots (d_0 - k + 1) = (2d)^k \prod_{i=0}^{k-1} (1 - \frac{i+1}{d}) = (2d)^k \left(1 - O\left(\frac{k^2}{d}\right)\right).$$

Also, the probability of a uniformly chosen random path in $\mathcal{P}$ meeting a specified vertex $v \in S$ is $\ell!/(2^\ell d_0(d_0 - 1) \ldots (d_0 - \ell + 1))$ where $\ell = \|v - v_0\|_1$. As $6 \leq \ell \leq k = o(d)$, this probability is decreasing exponentially with $\ell$, and so is at most $O(d^{-6})$. Thus at most a fraction $O(|S|d^{-6}) = O(d^{-1})$ of the paths in $\mathcal{P}$ meet $S$. Hence

$$\mathbb{E}(X_S | \mathcal{H}) \geq (1 - O(1/d))|\mathcal{P}|(1 - o(1))(p\alpha_{d,2})^k = (1 - o(1))(2\alpha_{d,2})^k.$$  \hfill (20)
Now we find an upper bound on the second moment of $X_S$. Fix $P_0 = (v_0, \ldots, v_k) \in \mathcal{P}$ avoiding $S$ and consider $\mathbb{E}(I_P \mid I_{P_0} = 1 \text{ and } \mathcal{H})$. If $I_P = I_{P_0} = 1$ and $P = (u_0, \ldots, u_k)$, then there exists an $r$ such that $u_i = v_i$ for $i \leq k - r$ and $u_i \neq v_i$ for $i > k - r$. (Recall that $I_{P_0} = 1$ implies there is a unique open path from $v_0$ to $v_k$, so $P$ cannot rejoin $P_0$ after they have separated.) Any two distinct vertices in $\mathbb{Z}^d$ have at most two common neighbors. Thus for $k - r < i < k$, $u_i$ has at least $2d - 8$ neighbors that are not neighbors of any other vertex of $P_0 \cup \{-e_d, u_0, \ldots, u_{i-1}\}$, as $u_i$ can only have a common neighbor with $u_{i-2}$, $v_{i\pm 2}$ or $v_i$. Let $\alpha$ be the probability that $u_i$ has at most $4$ open neighbors conditioned on $I_{P_0} = 1$, $\mathcal{H}$, the openness of $P$, and the state of the neighbors of $-e_d, u_0, \ldots, u_{i-1}$. Then $\alpha \leq \alpha_{d,8}$ since we know the two neighbors on $P$ are open, and so at most $2$ of the $2d - 8$ unconditioned neighbors must be open. As $u_i \neq v_i$ for $i > k - r$ we have

$$\mathbb{E}(I_P \mid I_{P_0} = 1 \text{ and } \mathcal{H}) \leq p^r \alpha_{d,8}^{-1}.$$ 

There are at most $(2d_0)^r \leq (2d)^r$ possible choices for $P$ having this intersection with $P_0$, and $p = 1/d$, so

$$\mathbb{E}(X_S \mid I_{P_0} = 1 \text{ and } \mathcal{H}) \leq \frac{1}{\alpha_{d,8}} \sum_{r=0}^k (2d \alpha_{d,8})^r \leq \frac{(2 \alpha_{d,8})^k}{\alpha_{d,8}(1 - \frac{1}{2 \alpha_{d,8}})} = \frac{(2 \alpha_{d,8})^k}{\alpha_{d,8} - \frac{1}{2}}. \quad (21)$$

Now, conditioned on $\mathcal{H}$, $\mathbb{E}(X_S^2 \mid \mathcal{H}) = \sum_{r=0}^k \mathbb{E}(I_P I_{P_0} = \sum_{P_0} \mathbb{E}(X_S \mid I_{P_0} = 1) p^r (I_{P_0} = 1)$. Thus combining (21) and (20) we have

$$\mathbb{E}(X_S^2 \mid \mathcal{H}) \leq \frac{(2 \alpha_{d,8})^k}{\alpha_{d,8} - \frac{1}{2}} \mathbb{E}(X_S \mid \mathcal{H}) \leq \frac{1 + o(1)}{\alpha_{d,8} - \frac{1}{2}} \left(\frac{\alpha_{d,8}}{\alpha_{d,2}}\right)^k \mathbb{E}(X_S \mid \mathcal{H})^2 = \frac{1 + o(1)}{\alpha_{d,8} - \frac{1}{2}} \mathbb{E}(X_S \mid \mathcal{H})^2,$$

where we have used $k = o(\sqrt{d})$ and $\alpha_{d,8}/\alpha_{d,2} = 1 + O(1/d)$. Now $k = \lceil 40 \log d \rceil$, so for sufficiently large $d$, $\log \mathbb{E}(X_S \mid \mathcal{H}) \geq (40 \log(2 \alpha_{\infty}) + o(1)) \log d \geq 12.1 \log d$, so $\mathbb{E}(X \mid \mathcal{H})/d^{12} \to \infty$ as $d \to \infty$.

For any non-negative random variable $X$ and $\varepsilon > 0$ we have $\mathbb{E}(X I_{(X > \varepsilon \mathbb{E}X)}) \geq (1 - \varepsilon)\mathbb{E}X$, and by Cauchy-Schwarz $\mathbb{E}(X^2 I_{(X > \varepsilon \mathbb{E}X)}) \mathbb{P}(X > \varepsilon \mathbb{E}X) \geq \mathbb{E}(X I_{(X > \varepsilon \mathbb{E}X)})^2$. Hence $\mathbb{P}(X > \varepsilon \mathbb{E}X) \geq (1 - \varepsilon)^2(\mathbb{E}X)^2/\mathbb{E}(X^2)$. Thus

$$\liminf_{d \to \infty} \mathbb{P}(X_S > d^{12} \mid \mathcal{H}) \geq \alpha_{\infty} - \frac{1}{2} > 0.1766.$$ 

The result follows.

Proof of Theorem 7. The fact that $\mathbb{Z}^d$ does not 3-percolate follows from Theorem 1 for all $d \geq 3$. Thus it remains to prove that for sufficiently large $d$, $\mathbb{Z}^d$ does 4-percolate.

We couple the process with one that dominates an independent oriented site percolation on $\mathbb{Z}^8$. An oriented site percolation is one where one requires an infinite path of open
sites in which each step of the path is obtained by increasing one of the coordinates of the point. Each site $x = (x_1, \ldots, x_8) \in (\mathbb{Z}_{>0})^8$ will correspond to a $d_0$-dimensional subspace $H_x$ of $\mathbb{Z}^d$ where the last 8 coordinates are fixed to be the coordinates of $3x$. Since the distance in $\mathbb{Z}^d$ between these regions is always at least 3, they are independent in the interference percolation model. The coupling will proceed one site at a time, processing all sites in the layer $\sum_{i=1}^8 x_i = \ell$ before starting with sites in the next layer $\sum_{i=1}^8 x_i = \ell + 1$. The order in which the sites within one layer are processed is not important. At each (reachable) site $x$, we shall define two events. The event $\mathcal{F}_x$ will be called the event that $x$ is pre-open. The event $\mathcal{F}_x \subseteq \mathcal{F}_x'$ will be called the event that $x$ is open. If there is an oriented path of open sites in $\mathbb{Z}^8$, i.e., sites $x$ for which $\mathcal{F}_x$ holds, then an infinite path will occur in the original interference percolation model. Also, conditioned on all previous sites, $\mathcal{F}_x$ will occur with probability $1 - o(1)$ and $\mathcal{F}_x$ will occur with probability at least $0.1766 - o(1)$. Every pre-open site $x$ will have an origin $O_x \in H_x$ and a predecessor $O_x^- = O_x - e_{d_x}$ defined, where $d_x > d_0$. If $x$ is pre-open then $O_x$ and $O_x^-$ will be open and the only open neighbors of $O_x^-$ will be $O_x$ and $O_x - 2e_{d_x}$. There will also be a set $S_x \subseteq H_x$ of bad vertices, where $|S_x| \leq d^5$ and all vertices of $S_x$ are at distance at least 6 from $O_x$. To start with, when $x = 0$, $S_x = \emptyset$ and $O_x$ and $O_x^- = O_x - e_{d_x}$ will be any vertices in $H_x = H_0$ satisfying these conditions. Note that as $H_x$ is infinite, such vertices will exist almost surely.

Now suppose $x \in \mathbb{Z}^8$ is pre-open. Then with probability at least $0.1766 - o(1)$, $X_{S_x} \geq d^{12}$. The event $\mathcal{F}_x$ will be the event that this holds (and of course that $x$ is pre-open).

Assume this occurs, so there are at least $d^{12}$ points in $H_x$ joined to $O_x$ by open paths. Each such point can be within distance 6 of at most $(2d)^6$ others. Hence there is a set $Y = \{y_1, y_2, \ldots, y_N\}$ of at least $d^{12}/(2d)^6 = d^5/2^6$ endpoints $y_i \in H_x$ of paths counted by $X_{S_y}$, each at least distance 7 from the others. Indeed, $Y$ may be constructed greedily by taking $y_i$ to be any such vertex at distance at least 7 from $\{y_1, \ldots, y_{i-1}\}$. In fact, points of $Y$ must be at distance at least 8 apart since they are all distance $k$ from $O_x$ and $\mathbb{Z}^d$ is bipartite. Consider one such $y_i$ and fix a direction $d' = d_0 + j$, $j \in \{1, \ldots, 8\}$. Then with probability $p^3 = d^{-3}$ we can extend the path from $y_i$ three steps in the positive $d'$-direction via open vertices $y_i + te_{d'}, t = 1, 2, 3$, to arrive at a vertex of $H_{x'}$, $x' = x + e_j$. The event $X_{S_y} \geq d^{12}$ depends only on the status of vertices within distance $k$ of $O_x$, and those points that are neighbors of $y_i + te_{d'}$ and are within distance $k$ of $O_x$ are assumed to be closed, except for those vertices along the open path. Indeed, there are no such neighbors for $t = 2$, the neighbors for $t = 1$ are closed by event $\mathcal{G}$ of the definition of $I_P$ above, and the only neighbor for $t = 0$ is the predecessor vertex along the open path. Thus with probability at least $q^{3(2d'-2)} = \Theta(1)$, $y_i + te_{d'}$ has no open neighbors except for those on the path for $t = 0, 1, 2$. Note that we make no assumption on the neighbors of $y_i + 3e_{d'}$. Consider the set of directions $d' = d_0 + j$ for which $H_{x'}$, $x' = x + e_j$, has not yet been ‘reached’, that is, no vertex $x' - e_k$, $k \neq j$, has previously been declared open. There are at most 8 such directions. Pick disjoint subsets $Y_j$ of $Y$ for each such
direction with \(|Y_j| = d^4/2 \gg d^3\). Then with probability \(1 - o(1)\) one can find \(y_{ij} \in Y_j\) so that \(y_{ij}\) has such a path in direction \(d_0 + j\). This will be the event \(F_{x'}\) that \(x' = x + e_j\) is pre-open. For each \(y_i \in Y_j\) that failed, place \(y_i + 3e_d\) and all its neighbors in the set \(S_{x'}\) of bad vertices assigned to the vertex \(x'\). At most \((1 + (2d - 16))d^4/2 \leq d^6\) such bad vertices will be generated. We define \(O_{x'} = y_{ij} + 3e_{d_0 + j}\) and \(O_x = y_{ij} + 2e_{d_0 + j}\). The only elements of \(H_{x'}\) for which we have any information about their neighbors all lie in \(\{O_x\} \cup S_{x'}\) and every point of \(S_{x'}\) is at distance at least 6 (actually at least 7) from \(O_x\). Thus when determining the openness of \(x'\) we shall not use any information about the status of vertices already encountered, except for those guaranteed by the pre-openness of \(x'\). As the percolation in \(\mathbb{Z}^8\) is oriented, paths out of \(H_{x'}\) will not have neighbors from previous steps. Two paths out of \(H_x\) do not share neighbors as they originate from different \(y_i \in Y\). Also note that \(H_{x'}\) is given just one chance to be pre-open. If this fails we do not try to construct paths from other vertices to \(H_{x'}\).

It is clear that for any vertex \(x \in \mathbb{Z}^8\) that is reachable from the origin by an open path in the oriented percolation defined above, there is some vertex in \(H_x\) that is reachable from \(O_0\) in the interference percolation model \((\mathbb{Z}^d)^{\leq 4}_p\). Thus the existence of an infinite open directed path in \(\mathbb{Z}^8\) implies the existence of an infinite path in \((\mathbb{Z}^d)^{\leq 4}_p\). As all (reachable) sites in \(\mathbb{Z}^8\) are open with probability at least \(0.1766 - o(1)\), even conditioned on the state of all previous sites, the open sites stochastically dominate an independent oriented site percolation model on \(\mathbb{Z}^8\) with site probability \(0.1766 - o(1)\). The result now follows from the following lemma.

**Lemma 13.** If \(p_c\) is the critical probability for oriented site percolation in \(\mathbb{Z}^8\), then \(p_c < 0.1735\).

**Proof.** Following the proof in [2] we have \(p_c \leq \rho_8\), where \(\rho_d\) is defined as the probability that two uniformly chosen random paths from the origin in \(\mathbb{Z}^d\) meet at a vertex other than the origin. (The proof in [2] is for bond percolation, but a trivial modification gives the above result for site percolation.) We estimate \(\rho_d\) by fixing a path \(P_0 = (v_0, v_1, \ldots)\) and calculating the probability that another path \(P = (u_0, u_1, \ldots)\) meets this at \(u_i = v_i\), \(i > 0\). Then

\[
\rho_d = \sum_{k=1}^{\infty} \mathbb{P}(u_k = v_k \text{ and } u_i \neq v_i, 0 < i < k).
\]

Now \(\mathbb{P}(u_k = v_k \mid P_0) = d^{-k} k! k_1k_2\ldots k_d!\) where \(v_k = (k_1, k_2, \ldots, k_d)\) and \(\sum_{i=1}^{d} k_i = k\). This is maximized when the \(k_i\) are as equal as possible. For \(i = 1, 2\) we have exact bounds of \(\mathbb{P}(u_1 = v_1) = d^{-1}\) and \(\mathbb{P}(u_2 = v_2, u_1 \neq v_1) = d^{-2} - d^{-3}\). Thus

\[
\rho_d \leq d^{-1} + d^{-2} - d^{-3} + \sum_{k=3}^{\infty} \frac{d^{-k}k!}{k_1!k_2!\ldots k_d!}.
\]

This series converges for \(d \geq 4\), and for \(d = 8\) gives \(p_c \leq \rho_8 < 0.1735\). \(\square\)
References

