Pair dominating graphs

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Abstract

An oriented graph dominates pairs if for every pair of vertices $u, v$, there exists a vertex $w$ such that the edges $\vec{wu}$ and $\vec{wv}$ both lie in $G$. We construct regular oriented triangle free graphs with this property, and thereby we disprove a conjecture of Myers. We also construct oriented graphs for which each pair of vertices is dominated by a unique vertex.

1 Introduction

Let $G$ be a digraph. We say that $G$ is 2-dominating or that it dominates pairs if for every pair $v_1, v_2 \in V(G)$ there exists $u \in V(G)$ such that $\vec{u}v_1, \vec{u}v_2 \in E(G)$. More generally, we say $G$ is $r$-dominating or that it dominates $r$-tuples if for every $r$-tuple $v_1, \ldots, v_r \in V(G)$ there exists $u \in V(G)$ such that $\vec{u}v_1, \ldots, \vec{u}v_r \in E(G)$. We say $G$ dominates pairs (or $r$-tuples) uniquely if the vertex $u$ is unique.

Let $g$ be the (directed) girth of $G$. If $g \geq 3$ then $G$ is an oriented graph, i.e., for each $u, v \in V(G)$, at most one of the edges $\vec{uv}$, $\vec{vu}$ lies in $E(G)$. We will be mostly interested in the case when $G$ is an oriented graph. For any vertex $v \in V(G)$, write $\Gamma^+(v) = \{w : \vec{w}v \in E(G)\}$ for the vertices dominated by $v$.

Myers [10] conjectured that every 2-dominating oriented graph contains a triangle. One of our aims is to give an infinite family of counterexamples to this conjecture. Myers was led to his conjecture by trying to prove a conjecture of Seymour (quoted by Dean and Latka [5]) saying that every oriented graph contains a vertex $v$ such that $|\Gamma^{++}(v)| \geq 2|\Gamma^+(v)|$, where $w \in \Gamma^{++}(v)$ iff $w$ is dominated by some vertex in $\Gamma^+(v) \cup \{v\}$.

The special case of Seymour’s conjecture for tournaments, Dean’s conjecture (see [5]), was proved by Fisher [6], and then a simpler proof was given by Havet and Thomassé [9]. Also, the special

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case of Seymour’s conjecture for circulant oriented graphs follows from the Cauchy-Davenport theorem (see [3], [4], [7], [8]) that for \( S \subseteq \mathbb{Z}_n \) we have \(|S + S| \geq \min\{n, 2|S| - 1\}\). A circulant oriented graph has vertex set \( \mathbb{Z}_n \) and its edges are given by a set \( S \subseteq \mathbb{Z}_n \setminus \{0\} \): a vertex \( a \) dominates a vertex \( b \) iff \( b - a \in S \). Our counterexamples to the conjecture of Myers are also circulant oriented graphs, i.e., we shall find sets \( S \subseteq \mathbb{Z}_n \setminus \{0\} \) such that \( S - S = \mathbb{Z}_n \), \( 0 \notin S + S \), and \( 0 \notin S + S + S \). We leave open the question whether \( S - S = \mathbb{Z}_n \) implies that \( S + S + S + S = \mathbb{Z}_n \), and we do not even know whether there is a \( k \) such that if \( S - S = \mathbb{Z}_n \) then the \( k \)-fold sum of \( S \) with itself is the entire \( \mathbb{Z}_n \).

Another of our aims in this paper is to show that there are infinitely many uniquely 2-dominating graphs. As we shall see, these are oriented graphs \( G \) such that the collection of out-sets \( \Gamma^+(v) \), \( v \in V(G) \), is the set of lines of a projective plane with point set \( V(G) \), and so is the collection of in-sets \( \Gamma^-(v) \). Another of the problems we leave open is whether there are triangle-free uniquely 2-dominating graphs. We shall show that the examples we construct all have oriented triangles.

## 2 Sum sets and difference sets

As stated above, we shall consider circulant digraphs obtained by taking \( V(G) = \mathbb{Z}_N \), the integers mod \( N \), and letting \( uvw \in E(G) \) iff \( v - u \in S \) for some suitably chosen set \( S \subseteq \mathbb{Z}_N \setminus \{0\} \). For the graph to be an oriented graph dominating pairs we need

\[
\begin{align*}
\text{S1.} & \quad S - S = \mathbb{Z}_N, \\
\text{S2.} & \quad 0 \notin S + S,
\end{align*}
\]

where \( S \pm S = \{a \pm b : a, b \in S\} \). In general, for the (directed) girth to be \( > k \) we need the \( r \)-fold sums \( S + S + \cdots + S \) not to contain 0 for all \( r \leq k \). If \( S \) is a set and \( n \in \mathbb{Z} \), write \( nS = \{nx : x \in S\} \).

**Lemma 1** \( \min_{S-S=\mathbb{Z}_N} |S + S + S| = o(N) \) as \( N \to \infty \).

**Proof.** Let \( T = \{2, 3, 11, 14, 17, 19, 21\} \). Then \( T - T = \{-19, \ldots, 19\} \setminus \{\pm 13\} \) and \( T + T + T \subseteq \{6, \ldots, 9\} \cup \{15, \ldots, 63\} \setminus \{29, 58\} \). Pick \( m \) minimal so that \( N \leq 24(29)^m \) and let \( S_m = T + 29T + (29)^2T + \cdots + (29)^mT \). First we prove by induction on \( m \) that \( S_m - S_m \supseteq \{-12(29)^m, \ldots, 12(29)^m\} \). This is clearly true for \( m = 0 \), so assume \( m > 0 \). Now \( S_m = 29S_{m-1} + T \), so \( S_m - S_m = 29(S_{m-1} - S_{m-1}) + (T - T) \). Pick \( x \) with \(|x| \leq 12(29)^m \). For all such \( x \), we can write \( x = 29x' + x'' \) with \(|x'| \in \{0, 12\} \cup \{14, 16\} \) and \(|x''| \leq 12(29)^{m-1} \). But \( x' \in S_{m-1} - S_{m-1} \) and \( x'' \in T - T \). As a consequence, \( S_m - S_m \) contains every residue class mod \( N \). Now consider the set \( S_m + S_m + S_m \). Let \( x \in S_m + S_m + S_m \) and write \( x \) in base 29, \( x = \sum a_i (29)^i \), \( a_i \in \{0, \ldots, 28\} \). We shall show that it is impossible that \( a_i = 12 \) and \( a_{i+1} = 1 \). Since \( x \in S_m + S_m + S_m \), \( x = \sum b_j (29)^j \), with \( b_j \in T + T + T \). Since \( b_i \leq 63 \), \( \sum_{j=0}^{i-1} b_j (29)^j < 3(29)^i \). Hence \( b_i \) must be 12, 11, or 10 mod 29. The only such \( b_i \) are 41, 40, and 39. But then \( b_{i+1} \equiv 0 \) mod 29, a contradiction. Since no pair
$(a_i, a_{i+1})$ can be $(12, 1)$ for any $i$, the number of elements in $S_m + S_m + S_m$ is $o((29)^{m+1}) = o(N)$. Hence if we let $S$ be the set of reductions of elements of $S_m$ mod $N$, then the number of elements in $S + S + S$ is also $o(N)$.

**Corollary 2** For all sufficiently large $N$ there is an oriented graph on $N$ vertices which dominates pairs and is (oriented) triangle-free.

**Proof.** Take $N$ large enough so that the $S$ given by the previous lemma satisfies $|S + S + S| < \frac{N}{12}$. Clearly $|S + S| \leq |S + S + S|$, so $|2(S + S + S) \cup 3(S + S)| < \frac{2N}{5}$, where for a set $T$ and $n \in \mathbb{Z}$, $nT = \{nx : x \in T\}$. Hence there are 6 consecutive elements mod $N$ that do not lie in $2(S + S + S) \cup 3(S + S)$. At least one of these will be divisible by 6 in $\mathbb{Z}_N$, say $6c$, and by replacing $S$ by $S - c$ we can ensure that $0 \notin 2(S + S + S) \cup 3(S + S)$. Then $0 \notin S + S$ and $0 \notin S + S + S$. This then gives an oriented graph as above which has no oriented triangles.

For $N = 29$ we can take $S$ to be the $T$ defined in Lemma 1. This is the smallest example we know of an oriented triangle-free graph that dominates pairs. There are several other constructions of such graphs. We list three such constructions.

**Blowing up vertices.**

Take any example of an oriented triangle free graph that dominates pairs (such as the above example on 29 vertices) and replacing one or more vertices by independent sets of vertices to give an example for larger $N$. This shows that examples exist for all $N \geq 29$. In general the graph constructed will not be a circulant graph.

**A simple explicit construction.**

Let $n \geq 8$ be an integer and let $S = \{1, 2, \ldots, n-2\} \cup \{n, 2n, 2n+1, 1-2n\}$. If $5n+3 \leq N \leq 6n-5$ then this gives an example on $\mathbb{Z}_N$. Note that such examples exist for all $N \geq 63$.

**Base $b$ expansion method.**

Choose $b, k > 1$ and let $N = b^k - 1$. For $a \in \mathbb{Z}_N$, considered $a$ as an integer in the range $0, \ldots, N-1$ and write $a$ in base $b$, $a = \sum_{i=0}^{k-1} a_i b^i$, $a_i \in \{0, \ldots, b-1\}$. Let $S$ be the set of $a$ for which $0 < \sum_{i=0}^{k-1} a_i < k(b-1)/3$. If $b$ and $k$ are sufficiently large then this also gives an example.

Although these constructions are simpler than that given by Lemma 1, we consider Lemma 1 to be of independent interest and pose the following.

**Question 1** Does there exist an $N$ and a set $S \subseteq \mathbb{Z}_N$ such that $S - S = \mathbb{Z}_N$, but $S + S + S \neq \mathbb{Z}_N$?

In case the answer to this question is in the affirmative, is it true that for every $k \geq 3$ there exist an $N$ and a set $S \subseteq \mathbb{Z}_N$ such that $S - S = \mathbb{Z}_N$, but the $k$-fold sum of $S$ with itself is not the whole of $\mathbb{Z}_N$?
3 Unique domination of pairs

Lemma 3 Suppose we are given a set of points \(P = \{p_1, \ldots, p_n\}\) and lines \(\{l_1, \ldots, l_m\}\), \(l_i \subseteq P\), with \(m \leq n\) such that every pair of points lie in a unique line. Then either

(a) there is a line containing all the points and all other lines have cardinality \(\leq 1\); or

(b) there is a line containing \(n - 1\) points and all other lines consist of one point from this line and the \(n\)th point; or

(c) \(n = d^2 + d + 1\), the points and lines form a projective plane of order \(d \geq 2\).

Proof. If two lines intersect in at least two points then these two points would not lie in a unique line. Hence the intersection of two lines contains at most one point. Assume \(l_1\) is the line with the largest number of points, and let \(|l_1| = a + 1\). If \(a + 1 = n\) then all the other lines can have at most 1 point and we are in case (a). Now assume \(a + 1 < n\) so there are some points not in \(l_1\). Then there must be lines that contain a point of \(l_1\) and a point not in \(l_1\). Let \(b + 1\) be the maximum size of such a line, say \(l_2\), and assume \(l_2\) intersects \(l_1\) at \(p\). We shall bound the number of lines \(m\). Each pair of points, not equal to \(p\), one from \(l_1\) and one from \(l_2\), specify a unique line, and all these lines are distinct. There are \(ab\) such lines none of which contain \(p\). The number of lines containing \(p\) is at least \(n - a - 1\) since these partition the \(n - 1\) points not equal to \(p\), and apart from \(l_1\) they all contain at most \(b\) points not equal to \(p\). Hence

\[
ab + 1 + \frac{n - a - 1}{b} \leq m \leq n. \tag{1}
\]

Rearranging gives \(a(b^2 - 1) \leq (n - 1)(b - 1)\). Assume now that \(b > 1\). Then

\[
n - 1 \geq a(b + 1). \tag{2}
\]

The number of lines other than \(l_1\) going through each point of \(l_1\) is at least \(\frac{n - a - 1}{b}\), and these lines are all distinct. Hence

\[
\frac{n - a - 1}{b}(a + 1) + 1 \leq m \leq n. \tag{3}
\]

Thus

\[
(n - 1)(a + 1 - b) \leq a(a + 1). \tag{4}
\]

Substituting inequality (2) into (4) gives

\[
(n - 1)(a + 2)a - (n - 1)^2 \leq a^2(a + 1) \tag{5}
\]

or, rearranging,

\[
(n - 1 - a)(n - 1 - a - a^2) \geq 0. \tag{6}
\]

The case \(n - 1 - a \leq 0\) implies that \(l_1\) contains all the points. Hence we may assume \(n - 1 - a > 0\). Thus

\[
n - 1 \geq a(a + 1). \tag{7}
\]
This together with equation (4) gives $b = a$ and $n = 1 + a + a^2$. In fact, there are at most $a$ lines other than $l_1$ through $p$, so all lines through $p$ must have $a + 1$ points. Since every point in now on a line with $a + 1$ points, we see that every line has $a + 1$ (or 0) points. If we had two non-intersecting lines with $a + 1$ points, then by considering lines meeting one point of the first and one point of the second, we would have a total of at least $(a + 1)^2 > n$ lines, a contradiction. Hence every two lines intersect in a single point, and the set of lines and points form a projective plane.

The only remaining case is when $b \leq 1$, so every line intersecting $l_1$ has at most 2 points. Hence for each pair of points, one in $l_1$ and one outside $l_1$, there is a unique line through the pair, and all such lines are distinct. This gives a total of $(a+1)(n-a-1) + 1$ lines. Thus $(a+1)(n-a-1) \leq n-1$, so

$$a(n-1) \leq a(a+1).$$

(8)

Hence $n \leq a + 2$ and we are in case (b).

**Corollary 4** If $G$ is an oriented graph that dominates pairs uniquely and $|V(G)| > 1$ then the sets $\Gamma^+(v)$ form the lines of a projective plane on $V(G)$.

**Proof.** Every pair of points is uniquely dominated, so lies in a unique line $\Gamma^+(v)$. The number of lines is the same as the number of points. Hence by Lemma 3 we either have a projective plane, or one of the two special cases listed in that lemma. It is easy to see that the two special cases cannot give rise to an oriented graph.

It remains to show that such oriented graphs exist. For this we consider the known projective planes, given by the lines in a three-dimensional vector space over a finite field.

**Theorem 5** For all $q = p^n$, $p$ prime, $n \geq 1$, there exists an oriented graph of order $q^2 + q + 1$ that dominates pairs uniquely.

**Proof.** Let $F_q$ be the field with $q$ elements. Let $F_{q^3}$ be the (unique) cubic extension of $F_q$. Then we can regard $F_{q^3}$ as a 3-dimensional vector space over $F_q$, and we can therefore regard the projective plane over $F_q$ as $F_{q^3}^\times/F_q^\times$, where $F_{q^3}^\times$ and $F_q^\times$ are the set of non-zero elements of $F_{q^3}$ and $F_q$ respectively. The lines of this projective plane correspond to 2-dimensional $F_q$-subspaces of $F_{q^3}$. Recall that the trace map $\text{Tr} : F_{q^3} \to F_q$ is a surjective $F_q$-linear map. For $\alpha \in F_{q^3}^\times$ (to be determined) let $G_\alpha$ be the following graph.

G1. $V(G_\alpha) = F_{q^3}^\times/F_q^\times$,

G2. $E(G_\alpha) = \{uv : \text{Tr}(\alpha v/u) = 0\}$,

where the condition $\text{Tr}(\alpha v/u) = 0$ is independent of the choice of representatives of $u, v$ in $F_{q^3}^\times$. The set $\Gamma^+(u)$ corresponds to a 2-dimensional subspace of $F_{q^3}$, so is a line in the projective space.
If the lines given by \( u_1 \) and \( u_2 \) are the same, then the linear maps \( v \rightarrow \text{Tr}(\alpha v/u_1) \) have the same kernel. But this implies the maps are proportional, \( \text{Tr}(\alpha v/u_2) = \lambda \text{Tr}(\alpha v/u_1) = \text{Tr}(\lambda \alpha v/u_1) \) for all \( v \). By letting \( v \) run over a basis for \( \mathbb{F}_q^3 \), we see that \( \alpha/u_2 = \lambda \alpha/u_1 \), so \( u_1 = \lambda u_2 \), and \( u_1 \) and \( u_2 \) give the same vertex of \( G_\alpha \). The only remaining conditions concern the girth. These are implied by the following condition

\[ C_1. \text{ If } \text{Tr}(x) = \text{Tr}(y) = 0 \text{ then } xy \neq \alpha^2. \]

To see this, take \( x = y = \alpha \). Then \( \text{Tr}(\alpha v/v) \neq 0 \) so \( G_\alpha \) contains no loops. If \( u, v \in V(G_\alpha) \), take \( x = \alpha u/v, y = \alpha v/u \). Then the condition shows that we cannot have both \( uv \) and \( vu \) in \( E(G) \).

The result now follows from the following lemma.

**Lemma 6** For every \( q = p^n, p \text{ prime}, n \geq 1 \), there exists \( \alpha \in \mathbb{F}_q^3 \) such that condition \( C_1 \) above holds.

**Proof.** The kernel of the trace map is a 2-dimensional \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q^3 \). Let \( \{ \eta, \eta' \} \) be a basis for this subspace and let \( \gamma = \eta' / \eta \). Now \( \gamma \notin \mathbb{F}_q \), so \( \mathbb{F}_q(\gamma) = \mathbb{F}_q^3 \) and \( \{ 1, \gamma, \gamma^2 \} \) is a basis for \( \mathbb{F}_q^3 \) over \( \mathbb{F}_q \).

The map \( \mathbb{F}_q^3 \rightarrow \mathbb{F}_q; k \mapsto k + 1/k \) is not surjective since \( |\mathbb{F}_q^3| < |\mathbb{F}_q| \). Hence there is an element \( c \in \mathbb{F}_q \) not of the form \( k + 1/k \). Let \( \beta = \eta^2 + c\eta / \eta' \). Then \( \beta \) is not the product of two trace-free numbers. Indeed, \( (a_1 \eta + a_2 \eta')(b_1 \eta + b_2 \eta') = \beta \) then \( 1 + \gamma^2 + \gamma^2 = (a_1 b_1) + (a_1 b_2 + a_2 b_1) \gamma + (a_2 b_2) \gamma^2 \). But since \( \{ 1, \gamma, \gamma^2 \} \) is a basis over \( \mathbb{F}_q \), we get

\[ a_1 b_1 = 1, \quad a_2 b_2 = 1, \quad a_1 b_2 + a_2 b_1 = c. \] (9)

This implies \( c = k + 1/k \) where \( k = a_1/a_2 \in \mathbb{F}_q \), a contradiction.

If \( \lambda \) is not a square in \( \mathbb{F}_q \) then it is not a square in \( \mathbb{F}_q^3 \) (otherwise \( \mathbb{F}_q(\sqrt{\lambda}) \) would be a quadratic extension of \( \mathbb{F}_q \) lying in \( \mathbb{F}_q^3 \)). Hence some element of the form \( \lambda \beta, \lambda \in \mathbb{F}_q^3 \), will be a perfect square in \( \mathbb{F}_q^3 \), since if \( \beta \) is not a perfect square, we can take \( \lambda \) to be a non-square in \( \mathbb{F}_q^3 \). Now choose \( \alpha \) so that \( \alpha^2 = \lambda \beta \).

If \( q \equiv 2 \mod 3 \) we can take \( \alpha = 1 \) in the lemma. To see this, we note that the trace is the sum of the conjugates, \( \text{Tr}(x) = x + x^q + x^{q^2} \). If \( \text{Tr}(x) = \text{Tr}(1/x) = 0 \) then \( x + x^q + x^{q^2} = x^{q^2} + x^{q^2-q^q} + x = 0 \). Thus \( x^{q^2-2q+1} = 1 \). Thus the order of \( x \) in the group \( \mathbb{F}_q^3 \) divides \( \gcd(q^2 - 2q + 1, q^3 - 1) = (q-1) \gcd(q-1, q^2 + q + 1) = q-1 \). But then \( x^{q-1} = x, \) so \( x \in \mathbb{F}_q \). But then \( \text{Tr}(x) = 3x \neq 0 \), a contradiction.

It is worth noting that the graphs obtained above have a cyclic automorphism. Indeed, \( \mathbb{F}_q^3 \times \mathbb{F}_q \) is a cyclic group under multiplication of order \( q^3 - 1 \), so \( \mathbb{F}_q^3/\mathbb{F}_q^3 \times \mathbb{F}_q^3 \) has the structure of a cyclic group of order \( q^3 + q + 1 \). The condition \( \text{Tr}(\alpha u/v) = 0 \) is just the condition that the difference in this cyclic group lies in a certain set \( S \), so the graph can be described as a circulant graph on \( \mathbb{Z}_N \) where \( N = q^2 + q + 1 \) and \( |S| = q+1 \).
4 Unique domination of \( n \)-tuples

**Lemma 7** Suppose we are given a set of points \( P = \{p_1, \ldots, p_n\} \) and lines \( \{l_1, \ldots, l_m\} \), \( l_i \subseteq P \), with \( m \leq n \), \( n \geq r \geq 3 \), such that every \( r \)-tuple of points lie in a unique line. Then either

(a) there is a line containing all the points and all other lines have cardinality \( < r \); or

(b) \( m = n = r + 1 \) and the lines consist of all subsets of \( P \) of size \( n - 1 \).

**Proof.** If there is a line containing every point then we are in case (a), and this must occur if \( r = n \), so assume \( r < n \) and some point, \( p \), say, does not lie in every line. Let \( l_1, \ldots, l_k \) be the lines containing \( p \) and \( l_{k+1}, \ldots, l_m \) the lines not containing \( p \). Each \((r - 1)\)-tuple of points in \( \{p_1, \ldots, p_{n-1}\} \) lies in an \( l_i \) with \( i \leq k \) since adding \( p \) to this \((r - 1)\)-tuple gives an \( r \)-tuple which lies in a line \( l_i \), and \( i \leq k \) since this line contains \( p \). If the \((r - 1)\)-tuple lies in two such lines, we would have two lines containing this \((r - 1)\)-tuple and \( p \). Thus every \((r - 1)\)-tuple lies in a unique line \( l_1, \ldots, l_k \). Since \( k \leq n - 1 \) we can apply induction on \( r \). If one of the lines \( l_i \), \( i \leq k \) contains all the points of \( P \setminus \{p\} \) then it contains all of \( P \) and we are in case (a). In all other cases in Lemma 3 or by induction on \( r \) from Lemma 7, \( k = n - 1 \). Hence there is only one line \( l_n \) not containing \( p \). Pick any point \( p_i \neq p_n \) and some \((r - 1)\)-tuple of points containing \( p_i \) but not \( p_n \). This \((r - 1)\)-tuple must lie in some unique line \( l_j \), \( j \leq k \). Pick a point \( p_h \) not in \( l_j \). Then the \( r \)-tuple obtained by adding \( p_h \) can only lie in \( l_n \) and so \( l_n \) contains \( p_i \). Thus \( l_n = P \setminus \{p_n\} \) and every other line can contain at most \( r - 1 \) points from \( P \setminus \{p_n\} \). There are \( \binom{n-1}{r-1} \) \( r \)-tuples containing \( p_n \), but each line \( l_1, \ldots, l_{n-1} \) can only contain one of these. Thus \( \binom{n-1}{r-1} \leq n - 1 \). Since \( 3 \leq r < n \) we have \( r = n - 1 \) and the lines consist of all \((n - 1)\)-tuples of points. \hfill \Box

**Theorem 8** For \( r \geq 3 \), the only directed graph with \( |V(G)| \geq r \) that dominates \( r \)-tuples uniquely is the complete digraph on \( r + 1 \) vertices.

**Proof.** The lines \( \Gamma^+(v) \) satisfy the conditions of Lemma 7. However, \( v \notin \Gamma^+(v) \) so we cannot be in case (a). Thus we are in case (b) with \( |V(G)| = r + 1 \) and \( |\Gamma^+(v)| = r \) for all \( r \). Hence there is a directed edge from \( v \) to every other vertex, and the graph is the complete digraph on \( r + 1 \) vertices. \hfill \Box

**References**


