From Toeplitz operators to black holes, and beyond

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Hilbert presents 10 of a list of 23 problems, among which the one that became known as Hilbert's twenty first problem.

In its original formulation, it is the question of surjectivity of the monodromy map in the theory of Fuchsian systems.

The problem got the name Riemann–Hilbert problem (RHP) for its obvious relation to a note by Riemann (1857) where he expressed a similar question.
An $n \times n$ linear system of differential equations

$$\frac{d\Psi}{dx} = A(x) \Psi(x)$$

is Fuchsian if the $n \times n$ coefficient matrix $A(x)$ is a rational function whose singularities are simple poles.

To each Fuchsian system one can associate a monodromy group.

The question of whether there always exists a Fuchsian system with given poles and a given monodromy group is Hilbert's 21st question.
Proof of the existence of linear differential equations having a prescribed monodromic group

In the theory of linear differential equations with one independent variable \( z \), I wish to indicate an important problem one which very likely Riemann himself may have had in mind. This problem is as follows: To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group. The problem requires the production of \( n \) functions of the variable \( z \), regular throughout the complex \( z \)-plane except at the given singular points; at these points the functions may become infinite of only finite order, and when \( z \) describes circuits about these points the functions shall undergo the prescribed linear substitutions. The existence of such differential equations has been shown to be probable by counting the constants, but the rigorous proof has been obtained up to this time only in the particular case where the fundamental equations of the given substitutions have roots all of absolute magnitude unity. L. Schlesinger (1895) has given this proof, based upon Poincaré's theory of the Fuchsian zeta-functions. The theory of linear differential equations would evidently have a more finished appearance if the problem here sketched could be disposed of by some perfectly general method.

(English translation from 1902)
The RHP was later put into the context of analytic factorisation of matrix valued functions, involving singular integral equations (Plemelj 1908) and holomorphic vector bundles (Röhrl 1957).

In 1990, Bolibruch constructed a counter example, thus giving a negative answer to Hilbert's question.

By then, a powerful analytic method—the RH method—had been developed to study a great variety of other problems and the original RHP became a special case of a RHP.
What is then a Riemann–Hilbert problem?

\[ \begin{align*}
C^+ & \quad \text{or} \quad D^+ \\
\text{or} & \quad D^- \\
C^- & \quad \phi_+ \text{ analytic in } C^+/D^-
\end{align*} \]
Ex 1. To determine \( \Phi \) such that

\[ \Delta \Phi = 0 \text{ in } \Omega, \quad \Phi \text{ continuous in } \overline{\Omega} \cup \partial \Omega \]

(1) \quad \Phi = f \text{ on } \partial \Omega

(2) \quad \Phi = \frac{\Phi^+ + \Phi^-}{2}; \quad \text{define } \Phi^-(z) = \Phi^+\left(\frac{1}{z}\right), \text{ for } |z| > 1

\Phi^+ \text{ analytic in } \Omega^-

\Phi^- \text{ analytic in } \Omega^-

\Phi^- (t) = \Phi^+ (t) \text{ on } \partial \Omega \quad (t = \frac{1}{z})

From (1) and (2):

\[ \frac{\Phi^+ (t) + \Phi^- (t)}{2} = f(t) \text{ on } \partial \Omega \]

\[ \leftrightarrow \quad -\Phi^+ (t) = \Phi^- (t) - 2f(t) \]

\( q \) is the coefficient of the RHP
Ex 2. Convolution equations

\[ \int_{-\infty}^{+\infty} g(t) i(t - t) \, dt = \delta(t) \]  

- the response of a LTI system to an input is described by the convolution of the input signal and the impulse response of the system.

- cross-correlation (important in signal processing, probability, and statistics) is a convolution

\[ R_{f_1, f_2} = f_1(-t) \ast f_2(t) \]
\[ \int_{-\infty}^{+\infty} g(t-z) \, i_{R^+}(z) \, dz = i_{R^+}(t), \quad t > 0 \]

extending by zero to IR

\[ \int_{-\infty}^{+\infty} g(t-z) \, i_{R^+}(z) \, dz = i_{R^+}(t) + i_{R^-}(t), \quad t \in \mathbb{R} \]

\[ \mathcal{F} \]

\[ G(w) \quad I_+(w) = I(w) + O(w) \quad (w \in \mathbb{R}) \]

analytic in \( \mathbb{C}^+ \)

analytic in \( \mathbb{C}^- \)
The coefficient may be matricial:

**Vectorial RHP** \[ G \phi_+ = \phi_- + \varphi \]

\[
\begin{align*}
G & \in \mathbb{R}^{n \times n} \\
\phi_+ & \in \mathbb{R}^{n \times 1} \\
\phi_- & \in \mathbb{R}^{n \times 1} \\
\varphi & \in \mathbb{R}^{n \times 1}
\end{align*}
\]

**Matrix RHP** \[ G M_+ = M_- \]

\[
\begin{align*}
G & \in \mathbb{R}^{n \times n} \\
M_+ & \in \mathbb{R}^{n \times n} \\
M_- & \in \mathbb{R}^{n \times n}
\end{align*}
\]

\[ (f=0) \]

\[ G = M_- M_+^{-1} \quad \text{Factorisation RHP} \]

With some additional conditions on \( M_+ \), this is called a **Wiener-Hopf** or **Birkhoff** factorisation.
It is a remarkable, and unanticipated, fact that a great variety of problems in mathematics, applied mathematics, and physics can be rephrased as RHP.

Classical fields of application include diffraction theory, elastodynamics, aerodynamics, control theory.

More recent fields of application range from integrable PDE's of KdV type to exactly solvable quantum field and statistical mechanics models, orthogonal polynomials, matrix models.
Some key references:

integrable PDE's: Manakov, Shabat, Zakharov (1975-1979)


orthogonal polynomials and matrix models:

Fokas, Kitaev, Its (1991)

Bleher, Deift, Krichever, Zhou, Its, ...
(late 1990's)

Baik, Deift, Johansson (1999)
A STEEPEST DESCENT METHOD FOR OSCILLATORY RIEMANN-HILBERT PROBLEMS

P. DEIFT AND X. ZHOU

In this announcement we present a general and new approach to analyzing the asymptotics of oscillatory Riemann-Hilbert problems. Such problems arise, in particular, in evaluating the long-time behavior of nonlinear wave equations solvable by the inverse scattering method. We will restrict ourselves here exclusively to the modified Korteweg de Vries (MKdV) equation,

\[ y_t - 6y^2y_x + y_{xxx} = 0, \quad -\infty < x < \infty, \quad t \geq 0, \]

\[ y(x, t = 0) = y_0(x), \]

but it will be clear immediately to the reader with some experience in the field, that the method extends naturally and easily to the general class of wave equations solvable by the inverse scattering method, such as the KdV, nonlinear Schrödinger (NLS), and Boussinesq equations, etc., and also to "integrable" ordinary differential equations such as the Painlevé transcendent.

As described, for example, in [IN] or [BC], the inverse scattering method for the MKdV equation leads to a Riemann-Hilbert factorization problem for a $2 \times 2$ matrix valued function $m = m(\cdot; x, t)$ analytic in $\mathbb{C}\setminus\mathbb{R}$,

\[
\begin{align*}
m_{+}(z) &= m_{-}(z)\nu_{x, t}, & z \in \mathbb{R}, \\
m(z) &\to I \quad \text{as} \quad z \to \infty,
\end{align*}
\]

where

\[ m_{\pm}(z) = \lim_{\epsilon \downarrow 0} m(z \pm i\epsilon; x, t), \]

\[ \nu_{x, t}(z) \equiv e^{-i(4t\alpha^3 + xz)}\sigma_3 \nu(z)e^{i(4t\alpha^3 + xz)}\sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

and

\[ \nu(z) = \begin{pmatrix} 1 & -|r(z)|^2 \\ r(z) & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{r} \\ \bar{r} & 1 \end{pmatrix} \equiv b^{-1}b_{+}. \]

If $y_0(x)$ is in Schwartz space, then so is $r(z)$ and

\[ r(z) = -\bar{r}(-z), \quad \sup_{z \in \mathbb{R}} |r(z)| < 1. \]

From the inverse point of view, given $\nu(z)$, one considers a singular integral

\[ \int \nu(z) \phi(z) dz, \]

where $\phi(z)$ is a test function. The formula for the asymptotics is then given by

\[ y(x, t) = \frac{i}{\sqrt{2\pi}} \int \nu(z) \phi(z) dz, \quad z \to \infty. \]

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A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics

By Percy A. Deift*, Alexander R. Its*, and Xin Zhou*

1. Introduction

Let \( J = \bigcup_{k=0}^{n} J_k = \bigcup_{k=0}^{n} (a_k, b_k) \) be a union of \( n+1 \) disjoint intervals in \( \mathbb{R} \). In this paper we consider the asymptotic behavior as \( x \to \infty \) of the Fredholm determinant \( P_x = \det(1 - K_x) \), where \( K_x \) is the (trace class) operator with kernel \( K_x(z, z') = \frac{\sin x(z-z')}{\pi (z-z')} \) acting on \( L^2(J, dz) \). The determinant \( P_x \) is the probability of finding no eigenvalues in the union of intervals \( \frac{x}{\pi} = \bigcup_{k=0}^{n} \left( \frac{a_k}{\pi}, \frac{b_k}{\pi} \right) \) for a random Hermitian matrix chosen from the Gaussian Unitary Ensemble (GUE), in the bulk scaling limit with mean spacing 1 (see [M1]).

Random matrix theory was introduced to the theoretical physics community as a subject of intensive study by Wigner in his work on nuclear physics in the 1950’s. Since that time random matrix theory has developed into an extremely active area of mathematics and of physics, with connections to many subareas, including, in particular, solvable field theories, and more recently, questions in number theory such as the distribution of zeros of the Riemann zeta function on the critical line \( \text{Re} \ z = \frac{1}{2} \) (see [M1], and also [RS]). Decisive early contributions to the subject were made, in particular, by des Cloizeaux, Dyson, Gaudin, Mehta, Widom, and Wigner, and also by Jimbo, Miwa, Mori, and Sato. The classical reference for the subject is Mehta’s book [M1], to which we refer the reader for historical discussion and also for the basic derivations.

In the one interval case, \( n = 0 \), after rescaling and translation, we may assume \( J = (-1, 1) \). For this case, in 1973, des Cloizeaux and Mehta [dCM] showed that as \( x \to \infty \)

\[
\log P_x = -\frac{x^2}{2} - \frac{1}{4} \log x + c + o(1)
\]

for some constant \( c \). Their method involved the use of explicit formulae for the

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Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model

By Pavel Bleher and Alexander Its

Abstract

We derive semiclassical asymptotics for the orthogonal polynomials $P_n(z)$ on the line with respect to the exponential weight $\exp(-NV(z))$, where $V(z)$ is a double-well quartic polynomial, in the limit when $n, N \to \infty$. We assume that $\varepsilon \leq (n/N) \leq \lambda_\text{cr} - \varepsilon$ for some $\varepsilon > 0$, where $\lambda_\text{cr}$ is the critical value which separates orthogonal polynomials with two cuts from the ones with one cut. Simultaneously we derive semiclassical asymptotics for the recursive coefficients of the orthogonal polynomials, and we show that these coefficients form a cycle of period two which drifts slowly with the change of the ratio $n/N$. The proof of the semiclassical asymptotics is based on the methods of the theory of integrable systems and on the analysis of the appropriate matrix Riemann-Hilbert problem. As an application of the semiclassical asymptotics of the orthogonal polynomials, we prove the universality of the local distribution of eigenvalues in the matrix model with the double-well quartic interaction in the presence of two cuts.

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Double Scaling Limit in the Random Matrix Model: The Riemann-Hilbert Approach

PAVEL BLEHER

AND

ALEXANDER ITS
Indiana University–Purdue University Indianapolis

Abstract

We prove the existence of the double scaling limit in the unitary matrix model with quartic interaction, and we show that the correlation functions in the double scaling limit are expressed in terms of the integrable kernel determined by the \( \psi \) function for the Hastings-McLeod solution to the Painlevé II equation. The proof is based on the Riemann-Hilbert approach, and the central point of the proof is an analysis of analytic semiclassical asymptotics for the \( \psi \) function at the critical point in the presence of four coalescing turning points.

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1 Introduction

In this paper we are concerned with the double scaling limit in the unitary random matrix model with quartic interaction. The unitary random matrix model, or the unitary ensemble of random matrices, is defined by the probability distribution

\[
\mu_N(dM) = Z_N^{-1} \exp(-N \text{ tr } V(M))dM,
\]

\[
Z_N = \int_{\mathcal{H}_N} \exp(-N \text{ tr } V(M))dM,
\]

on the space \( \mathcal{H}_N \) of Hermitian \( N \times N \) matrices \( M = (M_{ij})_{1 \leq i, j \leq N} \), where in general \( V(M) \) is a polynomial of even degree with a positive leading coefficient, or even more generally, a real analytic function with some conditions at infinity. The basic case for the double scaling limit is the quartic matrix model when

\[
V(M) = \frac{t}{2} M^2 + \frac{g}{4} M^4, \quad g > 0,
\]

and we will consider this case only. By a change of variable one can reduce the general case to the one with \( g = 1 \), but we prefer to keep \( g \) because it is useful in some questions. The double scaling limit describes the asymptotics of correlation functions between eigenvalues in the limit when simultaneously \( N \to \infty \) and \( t \) approaches the critical value

\[
t_c = -2\sqrt{g}.
\]
Multi-critical unitary random matrix ensembles and the general Painlevé II equation

By T. CLAEYS, A.B.J. KUIJLAARS, and M. VANLESSEN

Abstract

We study unitary random matrix ensembles of the form

$$Z_{n,N}^{-1} \det M^{2\alpha} e^{-N \text{Tr} V(M)} dM,$$

where $\alpha > -1/2$ and $V$ is such that the limiting mean eigenvalue density for $n, N \to \infty$ and $n/N \to 1$ vanishes quadratically at the origin. In order to compute the double scaling limits of the eigenvalue correlation kernel near the origin, we use the Deift/Zhou steepest descent method applied to the Riemann-Hilbert problem for orthogonal polynomials on the real line with respect to the weight $|x|^{2\alpha} e^{-NV(x)}$. Here the main focus is on the construction of a local parametrix near the origin with $\psi$-functions associated with a special solution $q_\alpha$ of the Painlevé II equation $q'' = sq + 2q^2 - \alpha$. We show that $q_\alpha$ has no real poles for $\alpha > -1/2$, by proving the solvability of the corresponding Riemann-Hilbert problem. We also show that the asymptotics of the recurrence coefficients of the orthogonal polynomials can be expressed in terms of $q_\alpha$ in the double scaling limit.

1. Introduction and statement of results

1.1. Unitary random matrix ensembles. For $n \in \mathbb{N}$, $N > 0$, and $\alpha > -1/2$, we consider the unitary random matrix ensemble

$$Z_{n,N}^{-1} \det M^{2\alpha} e^{-N \text{Tr} V(M)} dM,$$

on the space of $n \times n$ Hermitian matrices $M$, where $V : \mathbb{R} \to \mathbb{R}$ is a real analytic function satisfying

$$\lim_{x \to \pm \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty.$$

Because of (1.2) and $\alpha > -1/2$, the integral

$$Z_{n,N} = \int |\det M|^{2\alpha} e^{-N \text{Tr} V(M)} dM$$
Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities

By Percy Deift, Alexander Its, and Igor Krasovsky

Abstract

We study the asymptotics in \( n \) for \( n \)-dimensional Toeplitz determinants whose symbols possess Fisher-Hartwig singularities on a smooth background. We prove the general nondegenerate asymptotic behavior as conjectured by Basor and Tracy. We also obtain asymptotics of Hankel determinants on a finite interval as well as determinants of Toeplitz+Hankel type. Our analysis is based on a study of the related system of orthogonal polynomials on the unit circle using the Riemann-Hilbert approach.

1. Introduction

Let \( f(z) \) be a complex-valued function integrable over the unit circle with Fourier coefficients

\[
f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \ldots.
\]

We are interested in the \( n \)-dimensional Toeplitz determinant with symbol \( f(z) \),

\[
D_n(f(z)) = \det(f_{j-k})_{j,k=0}^{n-1}.
\]

In this paper we consider the asymptotics of \( D_n(f(z)) \) as \( n \to \infty \) and of the related orthogonal polynomials as well as the asymptotics of Hankel, and Toeplitz+Hankel determinants in the case when the symbol \( f(e^{i\theta}) \) has a fixed number of Fisher-Hartwig singularities \([22],[34]\), i.e., when \( f(e^{i\theta}) \) has the following form on the unit circle \( C \):

\[
f(z) = e^{V(z)z} \prod_{j=0}^m (z - z_j)^{-2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad \theta \in [0,2\pi),
\]

for some \( m = 0, 1, \ldots, \) where

\[
z_j = e^{i\theta_j}, \quad j = 0, \ldots, m, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi,
\]

\[
g_{z_j,\beta_j}(z) \equiv g_{\beta_j}(z) = \begin{cases}
e^{i\pi\beta_j}, & 0 \leq \arg z < \theta_j, \\
e^{-i\pi\beta_j}, & \theta_j \leq \arg z < 2\pi,
\end{cases}
\]
Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach
The Riemann-Hilbert Problem and Integrable Systems

Alexander R. Its

In its original setting, the Riemann-Hilbert problem is the question of surjectivity of the monodromy map in the theory of Fuchsian systems.

An $N \times N$ linear system of differential equations

$$\frac{d\psi(\lambda)}{d\lambda} = A(\lambda)\psi(\lambda)$$

is called Fuchsian if the $N \times N$ coefficient matrix $A(\lambda)$ is a rational function of $\lambda$ whose singularities are simple poles. Each Fuchsian system generates, via analytic continuation of its fundamental solution $\psi(\lambda)$ along closed curves, a representation of the fundamental group of the punctured Riemann sphere (punctured at the poles of $A(\lambda)$) in the group of $N \times N$ invertible matrices. This representation (or rather its conjugacy class) is called the monodromy group of equation (1), and it is the principal object of the theory of Fuchsian systems. The question of whether there always exists a Fuchsian system with given poles and a given monodromy group was included by Hilbert in his famous list as problem number twenty-one. The problem got the name "Riemann-Hilbert" for its obvious relation to the general idea of Riemann that an analytic (vector-valued) function could be completely defined by its singularities and monodromy properties.

Subsequent developments put the Riemann-Hilbert problem into the context of analytic factorization of matrix-valued functions and brought to the area the methods of singular integral equations (Plemelj, 1908) and holomorphic vector bundles (Rohrlich, 1957). This result eventually in a negative (!) solution, due to Bolibruch (1989), of the Riemann-Hilbert problem in its original setting and to a number of deep results (Bolibruch, Kostov) concerning a thorough analysis of relevant solvability conditions. We refer the reader to the book of Anosov and Bolibruch [2] for more on Hilbert's twenty-first problem and the fascinating history of its solution (and for more details on the genesis of the name "Riemann-Hilbert").

Simultaneously, and to a great extent independently of the solution of the Riemann-Hilbert problem itself, a powerful analytic apparatus—the Riemann-Hilbert method—was developed for solving a vast variety of problems in pure and applied mathematics. The Riemann-Hilbert method reduces a particular problem to the reconstruction of an analytic function from jump conditions or, equivalently, to the analytic factorization of a given matrix- or scalar-valued function defined on a curve. Following a tradition that developed in mathematical physics, it is these problems, and not just the original Fuchsian one, that we will call Riemann-Hilbert problems. In other words, we are adopting a point of view according to which the Riemann-Hilbert (monodromy) problem is formally treated as a special case (although an extremely important one) of a Riemann-Hilbert (factorization) problem. The latter is viewed as an analytic tool, but one whose implementation is not at all algorithmic and which might use quite sophisticated and

\[ ^{1}\text{It should be mentioned that in the theory of boundary values of analytic functions the problem of reconstructing a function from its jumps across a curve is sometimes called the "Hilbert boundary-value problem". This adds even more subtlety to the origin of the name "Riemann-Hilbert problem".} \]
There are always two steps in the RH method:

I - Reducing the original problem to a RHP, i.e., to the problem of finding a sectionally analytic function in the complex plane having a prescribed jump across a curve.

This often means replacing a nonlinear original problem by a linear one.

II - Solving the RHP

⚠️ Scalar and matrixial RHP present very different levels of difficulty!
Wave diffraction by a half-plane with an obstacle perpendicular to the boundary

L.P. Castro \(^a\),\(^*,1\), D. Kapanadze \(^b\)

\(^a\) Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal
\(^b\) A. Razmadze Mathematical Institute, Tbilisi State University, Tbilisi, Georgia

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**Abstract**
We prove the unique existence of solutions for different types of boundary value problems of wave diffraction by a half-plane with a screen or a crack perpendicular to the boundary. Representations of the solutions are also obtained upon the consideration of some associated operators. This is done in a Bessel potential spaces framework and for complex (non-real) wave numbers. The investigation is mostly based on the construction of explicit operator relations, the potential method, and a factorization technique for a certain class of oscillating Fourier symbols which naturally arise from the problems.

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1. Introduction

The physical motivations behind the present study arise from the problem of acoustic and electromagnetic time-harmonic plane wave diffraction by a strip interacted with the boundary. In particular, we deal with boundary value problems for the Helmholtz equation, where the strip is located in the \(Oxz\)-plane (when adopting the Cartesian axes \(Oxyz\)) and perpendicular to \(y\)-axis – which may be viewed as a boundary of an obstacle. Throughout this work we assume that the material is invariant in the \(z\)-direction. Thus, in effect, the geometry of the problem is two-dimensional, which leads us
Wave diffraction by wedges having arbitrary aperture angle

L.P. Castro\textsuperscript{a,*}, D. Kapanadze\textsuperscript{b}

\textsuperscript{a} CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal
\textsuperscript{b} A. Razmadze Mathematical Institute, Tbilisi State University, Tbilisi, Georgia

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\textbf{A B S T R A C T}

The problem of plane wave diffraction by a wedge sector having arbitrary aperture angle has a very long and interesting research background. In fact, we may recognize significant research on this topic for more than one century. Despite this fact, up to now no clear unified approach was implemented to treat such a problem from a rigorous mathematical way and in a consequent appropriate Sobolev space setting. In the present paper, we are considering the corresponding boundary value problems for the Helmholtz equation, with complex wave number, admitting combinations of Dirichlet and Neumann boundary conditions. The main ideas are based on a convenient combination of potential representation formulas associated with (weighted) Mellin pseudo-differential operators in appropriate Sobolev spaces, and a detailed Fredholm analysis. Thus, we prove that the problems have unique solutions (with continuous dependence on the data), which are represented by the single and double layer potentials, where the densities are solutions of derived pseudo-differential equations on the half-line.

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1. Introduction

The problem of plane wave diffraction by wedge sectors counts already more than one century of research. Indeed, we may identify the classical works of Sommerfeld [67] and Poincaré [60] as the first ones where this type of problem was significantly tackled. There, the solution of the Helmholtz equation in an infinite wedge sector with Dirichlet and Neumann boundary conditions was studied by using the Sommerfeld integrals and separation of variables, respectively. Anyway, previous partial results can also be identified. This is the case of Macdonald [39] who already gave in 1895 a representation of the first and second Green’s functions (i.e., electrostatic and velocity potentials) of the potential equation for a wedge of an arbitrary aperture angle. In fact, this was first considered only for angles of the form $\pi/m$, where $m$ is a positive integer, and later (cf. [40]) Macdonald was able to obtain formulas for the two Green’s functions of the Helmholtz equation.

\* Corresponding author.
\textbf{E-mail addresses:} castro@ua.pt (L.P. Castro), david.kapanadze@gmail.com (D. Kapanadze).

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Convolution Type Operators with Symmetry in Bessel Potential Spaces

Luís Pinheiro de Castro and Frank-Olme Speck

Dedicated to Roland Duduchava on the occasion of his 70th birthday

Abstract. Convolution type operators with symmetry appear naturally in boundary value problems for elliptic PDEs in symmetric or symmetrizable domains. They are defined as truncations of translation invariant operators in a scale of Sobolev-like spaces that are convolutionally similar to subspaces of even or odd functionals. The present class, as a basic example, is closely related to the Helmholtz equation in a quadrant, where a possible solution is “symmetrically” extended to a half-plane. Explicit factorization methods allow the representation of resolvent operators in closed analytic form for a large class of boundary conditions including the two-impedance and the oblique derivative problems. Moreover they allow fine results on the regularity and asymptotic behavior of the solutions.

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Keywords. Convolution type operator, symmetry, factorization, boundary value problem, quadrant, diffraction, explicit solution, Sobolev space.

1. Introduction

Let \( r, s \in \mathbb{R} \). We consider operators of the form

\[
T = r_+ A_\Phi : H^{r,c}(\mathbb{R}) \to H^s(\mathbb{R}_+)
\]  

(1.1)

where \( r_+ \) denotes the restriction operator to \( \mathbb{R}_+ \), \( A_\Phi = \mathcal{F}^{-1} \Phi \cdot \mathcal{F} : H^r \to H^s \) stands for a convolution (translation invariant) operator that is invertible of order \( r - s \), i.e.,

\[
\lambda^{s-r} \Phi \in \mathcal{G}L^\infty(\mathbb{R})
\]  

(1.2)

where \( \lambda(\xi) = (\xi^2 + 1)^{1/2} \), \( \xi \in \mathbb{R} \), \( \mathcal{G}L^\infty(\mathbb{R}) \) denotes the group of invertible elements in \( L^\infty(\mathbb{R}) \) and \( \mathcal{F} \) denotes the one-dimensional Fourier transformation (we also call the Fourier symbol \( \Phi \) to be invertible of order \( r - s \) or briefly \( r - s \)-invertible...
There are always two steps in the RH method:

I - Reducing the original problem to a RHP, i.e., to the problem of finding a sectionally analytic function in the complex plane having a prescribed jump across a curve.

This often means replacing a nonlinear original problem by a linear one.

II - Solving the RHP

Scalar and matrixial RHP present very different levels of difficulty!
There are no general methods to solve matrix RHP; neither to obtain criteria of solvability, nor to obtain explicit solutions when they exist.

Alexander Its:

The RH (factorisation) problem is viewed as an analytical tool, but one whose implementation is not at all algorithmic and which might require the use of quite sophisticated and "custom-made" analytic ideas (depending on the particular setting of the factorisation problem).
1. The nonlinear steepest descent method for oscillatory RHP (Deift, Zhou, 1992) has been used with great success to analyse the asymptotic behavior of systems depending on a large parameter, such as time or space, or a small parameter, such as perturbation strength, which are formulated in terms of a RHP.

2. Many other applications, however, require clear and verifiable a priori criteria for solvability and truly explicit solutions to the RHP.
Considerable progress has been made in the last decade in explicit factorisation methods.


I. Spectral properties of truncated Toeplitz operators (TTO)

1. To help understand what a Toeplitz operator and a TTO are:
   \[ M_g : \mathbb{C} \rightarrow \mathbb{C} \ , \ z \rightarrow g z \ (g \in \mathbb{C}) \]
   \[ \mathbb{C} = \mathbb{R} \oplus i\mathbb{R} \ , \ P^+ : \mathbb{C} \rightarrow \mathbb{R} \ , \ P^- : \mathbb{C} \rightarrow i\mathbb{R} \]

Kernel of T:
\[ T x = 0 \iff g x \in i\mathbb{R} \]

Compress of \( M_g \) to \( \mathbb{R} \)
2. Toeplitz operators \( (T_g) \)

\[ M_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad \varphi \mapsto g \varphi \quad (g \in L^\infty(\mathbb{R})) \]

\[ L^2(\mathbb{R}) = H_2^+ \oplus H_2^- \quad H_2^+ = \mathcal{F} L^2(\mathbb{R}^+) \]

\[ p^\pm : L^2(\mathbb{R}) \rightarrow H_2^\pm \quad \text{projections} \]

\[ T_g \varphi_+ = p^+ g \varphi_+ \]

\( T_g : H_2^+ \rightarrow H_2^+ \) is the compression of \( M_g \) to \( H_2^+ \)
Kernels of Toeplitz operators can be naturally described in terms of RHP:

\[ \varphi_+ \in \ker T_g \iff P_g^+ \varphi_+ = 0 \iff g \varphi_+ = \varphi_- \]

\[ e \in \mathbb{H}_2^+ \]

\[ g \text{ is the symbol of the operator} \]

\[ \text{It can be scalar or maticial} \]

3. Truncated Toeplitz operators \((A^0_\theta)\) are compressions of \(M_\theta\) to certain subspaces of \(\mathbb{H}_2^+\) called model spaces, denoted by \(K_\theta\).
Ex 1: $k_{z^n} = \text{span} \{1, z, z^2, \ldots, z^{n-1}\}, \ z \in \mathbb{D}$

TTO in $k_{z^n}$ are represented with respect to this basis by $n \times n$ Toeplitz matrices.

\[
\begin{bmatrix}
  a_0 & a_{-1} & a_2 & a_{-3} \\
  a_1 & a_0 & a_{-1} & a_{-2} \\
  a_2 & a_1 & a_0 & a_{-1} \\
  a_3 & a_2 & a_1 & a_0
\end{bmatrix}
\]

Compression of multiplication by

$q = a_{-3} z^{-3} + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + q_1 z + q_2 z^2 + q_3 z^3$

to

$k_{z^4} = \text{span} \{1, z, z^2, z^3\}$

When $a_{-3} = a_1, \ a_{-2} = a_2, \ a_{-1} = a_3$ this is a circulant matrix.
Ex 2 Volterra integral operator in $L^2([0,1])$

$$V \varphi(x) = \int_0^x \varphi(t) \, dt, \quad 0 < x < 1$$

- Extending the functions in $L^2([0,1])$ by zero to $1 \mathbb{R}$ and applying the Fourier transform we get a TTO in the model space

$$K_{e^{ix}} = \mathcal{F} L^2([0,1])$$

- In this case

$$A_{g_{e^{ix}}} : K_{e^{ix}} \rightarrow K_{e^{ix}}$$

is the compression of multiplication by $g(x) = \frac{e^{ix} - 1}{ix}$ to $K_{e^{ix}}$
D. Sarason - Algebraic properties of truncated Toeplitz operators, 2007


While some natural operator theoretic questions about TTO (normal, self-adjoint, unitary?) are beginning to be answered, others, such as a description of their spectrum, remain open except for rather particular cases.

- TTO are equivalent after extension to Toeplitz operators with matrix symbols of a particular type.

- The study of the spectra of TTO is reduced to that of kernels of Toeplitz operators with matrix symbols, i.e., to RHP.

M. C. Câmara and J. R. Partington - Spectral properties of truncated Toeplitz operators by equivalence after extension
II. Constructing solutions to EFE

EFE are nonlinear PDE's describing spacetime deformations in the presence of mass/energy, in terms of a certain metric.

They are very difficult to solve in general, so our attention must concentrate on special classes of solutions exhibiting symmetries.

By focusing on solutions with a high enough number of isometries, EFE can be reduced to 2-dimensional field equations.
(1) \[ d(p \star A) = 0, \quad A = M^{-1} dM \]

\[ M = M(x) \quad \text{for} \quad x = (p, v) \]

* Hodge dual \( \star dp = -dv, \quad \star dv = dp, \quad \star^2 = -id \)

\[ d \quad \text{exterior derivative} \]

Following Breitenlohner-Maison's approach


we construct a **linear** system (lax pair) depending on an additional variable \( \xi \) (spectral parameter)

(2) \[ \xi (d + A)X = \star dX \]

which is overdetermined, in such a way that (1) appears as a compatibility condition for (2). It is thus an **integrable system** (Its)
Suppose that we have a solution $M(x)$ of the 2-dimensional eq.
of motion and that there is a solution $X$ to the linear system
\[ \mathcal{E}(d+A)X = *dX, \quad A = M^{-1}dM \]
satisfying certain regularity conditions, namely
$X(\tau, r, v)$ is analytic as a function of $\tau$ in a
neighborhood of the origin ($\tau=0$),
continuous up to the boundary,
$X^{-1}$ satisfying the same regularity
conditions.

[CCMN] M.C. Cámara, G.L. Cardoso, T. Mohaupt, S. Nampuri
A Riemann-Hilbert approach to rotating attractors, JHEP, 2017
Define

\[ M_0(\varepsilon, \nu) = X^*(-\frac{1}{\varepsilon}, \nu) \, M(\nu) \, X(\varepsilon, \nu) \]

where \( \# \) denotes an involutive anti-homomorphism such that \( M^*(\nu) = M(\nu) \).

The right-hand side of (3) is a Wiener-Hopf/Birkhoff factorisation.

The left-hand side can be written as

\[ M(w) \quad \text{with} \quad w = \nu + \frac{p}{2} \frac{1 - \varepsilon^2}{\varepsilon} \]

\[ \underline{\text{monodromy matrix}} \quad \underline{\text{spectral curve}} \]

\[ M^*(w) = M(w) \]
From RHP to EFE

Given $M(\omega)$ satisfying $M^*(\omega) = M(\omega)$, if

$$M(\omega) \mid_{\omega = \nu + \frac{p}{2} \frac{1-\varepsilon^2}{\varepsilon}} = M_-(\varepsilon, \nu) \quad M_+(\varepsilon, \nu)$$

where $\nu = (p, \nu)$ appear as a parameter in the factorisation and $\begin{cases} M_+^{\pm1} & \text{analytic and bounded for } |\varepsilon| < 1 \\ M_-^{\pm1} & \text{analytic and bounded for } |\varepsilon| > 1 \end{cases}$

then

$$M(\omega) \mid_{\omega = \nu + \frac{p}{2} \frac{1-\varepsilon^2}{\varepsilon}} = X^*(\frac{1}{\varepsilon}, \nu) \quad M(\nu) \quad X(\varepsilon, \nu),$$

and $X(0, \nu) = I$.

and $M(\nu) = \lim_{\varepsilon \to 0} M_-(\varepsilon, \nu)$ provides a solution to (1).
Solve the longstanding problem of constructing extremal black holes within the RH formulation.

By acting with elements of the so-called Geroch group on the monodromy matrices of certain "seed solutions", we obtain new solutions to EFE.

An example: Deforming the monodromy matrix corresponding to the solution that describes the near horizon region of an extremal black hole, we obtain a new solution - explicit, exhibiting unexpected properties.

Although the deformation of the monodromy matrix is linear in $\alpha$, the spacetime solution receives corrections of order $\alpha^2$.

As $\beta \to 0$, the deformation of the spacetime solution disappears - in this limit, it describes an attractor background.
Further work by Gabriel Cardoso and João Serra:

applying a Möbius transformation in \( w \) to a monodromy matrix corresponding to the near horizon solution of an extremal black hole, they obtained an interpolating solution between flat space-time and a black hole horizon, exhibiting dependence on the angular variable — and more!

JHEP, March 2018

New gravitational solutions via a Riemann–Hilbert approach
Lessons learnt

- The amount of information encoded in the spectral parameter $\tau$, which appears as an auxiliary complex variable alien to the original problem has yet to be fully understood.
- Factorisations may transform innocent looking deformations of the monodromy matrix into highly non-trivial deformations of the space-time solutions.
- There are "quasi-disjoint" communities who can benefit from solving RHP, and developments in one field often go unnoticed in another—there is much to gain in creating conditions for them to interact.