Algebraic reflexivity of spaces of analytic functions

Priyadarshi Dey

University of Memphis

pdey@memphis.edu

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Outline

1. Introduction
2. Definitions and Notations
3. Some results
4. Question: Is $S^p$ algebraic reflexive?
5. Proof of $S^p$ is algebraically reflexive
Introduction

**Problem:** Kaplansky’s question: When is a function locally given by a polynomial is a polynomial?

In this talk we will answer a similar question related to isometry of some space.

People who have mostly work on this Kadison, Larson, TSSRK Rao, Jamison, Fernanda Botelho, Molnar and more..
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Definitions and Notations

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- \( \mathbb{D} \) denotes the open unit disk in \( \mathbb{C} \).
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- \( \mathbb{C} \) denotes the set of all complex numbers.
- \( \mathbb{D} \) denotes the open unit disk in \( \mathbb{C} \).
- \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \).
**Definition (Isometry)**

Given two normed space $X$ and $Y$, we say a linear map $T : X \rightarrow Y$ an isometry if

$$\|Tx\| = \|x\| \text{ for every } x \in X$$

**Definition (Locally Surjective Isometry)**

A linear map $T$ from a Banach space $X$ into itself is called Locally Surjective Isometry if for each $a \in X$ there is a surjective isometry $T_a : X \rightarrow Y$ such that $T(a) = T_a(a)$

**Definition (Algebraic Reflexivity)**

A Banach space $X$ is called Algebraic Reflexive if any locally surjective isometry on $X$ is surjective.
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Definition (Algebraic Reflexivity)

A Banach space $X$ is called Algebraic Reflexive if any locally surjective isometry on $X$ is surjective.
A holomorphic function $h: \mathbb{C} \to \mathbb{C}$ is called an inner function if $|z| \leq 1$ on the unit disk and $\lim_{r \to 1^-} h(re^{i\theta})$ exists for almost all $\theta$ and its modulus is 1 a.e.
Definitions and Notations

**Definition (Inner function)**

A holomorphic function \( h: \mathbb{C} \mapsto \mathbb{C} \) is called an inner function if \( |z| \leq 1 \) on the unit disk and \( \lim_{r \to 1^-} h(re^{i\theta}) \) exists for almost all \( \theta \) and its modulus is 1 a.e.

**Definition (Conformal map)**

Let \( U \) and \( V \) be subsets of \( \mathbb{D} \). A function \( f: U \mapsto V \) is called conformal at a point \( u_0 \in U \) if it preserves angles between directed curves through \( u_0 \), as well as preserving orientation.
Theorem (Conformal maps on $\mathbb{D}$)

Let $\phi$ be a conformal map from $\mathbb{D}$ into $\mathbb{D}$. Then, $\exists \theta \in \mathbb{R}$ such that

$$\phi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha}z} \quad (\alpha \in \mathbb{D})$$
The space $S^p$

Definition (The Hardy Space, $H^p$)

For $p \geq 2$,

$$H^p := \left\{ f : \mathbb{D} \xrightarrow{\text{analytic}} \mathbb{D} : \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} < \infty \right\}$$
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The norm we consider on $S^p$ is given by

$$\|f\| = |f(0)| + \|f'||_p$$
Isometries of $H^p$

Theorem (F. Forelli, Can. J. Math. 16 (1964), 721-728)

Suppose that $p \neq 2$ and $T$ is a linear isometry from $H^p$ into $H^p$. Then there is a non constant inner function $\phi$ and a function $F \in H^p$ such that

$$Tf = F \cdot f(\phi)$$
The isometries of $S^p$ with $\|f\| = |f(0)| + \|f'\|_p$

Theorem (Into isometries of $S^p$, W.P. Novinger & D.M. Oberlin)

Let $T$ be a linear isometry of $S^p$ into $S^p$. Then there is a linear isometry $\tau$ of $H^p$ into $H^p$ and a unimodular complex number $\lambda$ such that

$$Tf(z) = \lambda[f(0) + \int_0^z \tau f'(\zeta)d\zeta]$$
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Corollary

Let $T$ be a linear isometry of $S^p$ into $S^p$ and $p \neq 2$. Then there is a non-constant inner function $\phi$ and a function $F$ in $H^p$ such that

$$Tf(z) = \lambda [f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta], z \in \mathbb{D}, f \in S^p$$
Onto isometries of $S^p$

Theorem (Onto isometries of $S^p$, W.P.Novinger & D.M.Oberlin)

Let $T$ be a linear isometry of $S^p$ onto $S^p$, $p \neq 2$. Then $\exists \lambda, \mu \in \mathbb{T}$ and a conformal map $\phi: \mathbb{D} \to \mathbb{D}$ such that

$$Tf(z) = \lambda[f(0) + \mu \int_0^z [\phi'(\zeta)]^{\frac{1}{p}} f'(\phi(\zeta)) d\zeta]$$
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Conversely, this equation defines an isometry $T$ of $S^p$ onto $S^p$.
Is $S^p$ algebraic reflexive?
Into isometries on $S^p$: $Tf(z) = \lambda [f(0) + \int_0^z \tau f'(\zeta) d\zeta]$

Onto isometries: $Tf(z) = \lambda [f(0) + \mu \int_0^z [\phi'(\zeta)]^{\frac{1}{p}} f'(\phi(\zeta)) d\zeta]$

**Theorem (F. Botelho, J. Jamison)**

*For each $p \neq 2$, $S^p$ is Algebraic reflexive.*
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**Proof.**

We have to prove any locally surjective isometry is surjective on $S^p$. 
Into isometries on $S^p$: $Tf(z) = \lambda [f(0) + \int_0^z \tau f'(\zeta) d\zeta]$

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**Theorem (F. Botelho, J. Jamison)**

*For each $p \neq 2$, $S^p$ is Algebraic reflexive.*

**Proof.**

We have to prove any locally surjective isometry is surjective on $S^p$. Let $T$ be a locally surjective isometry. Then by a previous theorem,

$$Tf(z) = \lambda [f(0) + \int_0^z F(\zeta) f'(\phi(\zeta)) d\zeta] \quad f \in S^p$$
Into isometries on $S^p$: $Tf(z) = \lambda [f(0) + \int_0^z \tau f'(\zeta) d\zeta]$

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$$Tf(z) = \lambda [f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta] \quad f \in S^p$$

$$= \lambda_f [f(0) + \mu_f \int_0^z [\phi_f'(\zeta)]^{\frac{1}{p}} f'(\phi_f(\zeta))d\zeta] \quad (1)$$
\[ \lambda[f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta] = \]
\[ \lambda_f[f(0) + \mu_f \int_0^z \left(\phi'_{f(\zeta)} \right)^{1/p} f'(\phi_{f(\zeta)})d\zeta] \]

**Proof.**

To prove, \( \lambda_f = \lambda \).
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Taking the derivative of (1), we get:

$$
\lambda \cdot F(z)f'(\phi(z)) = \lambda_f \mu_f [\phi_f'(z)]^{\frac{1}{p}} f'(\phi_f(z)) \ , \ \forall f
$$
\[
\lambda [f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta] = \\
\lambda f[0] + \mu f \int_0^Z [\phi_f'(\zeta)]^{1/p} f'(\phi_f(\zeta))d\zeta
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Now, from (1),

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\lambda f(0) + \int_0^Z \lambda F(\zeta)f'(\phi(\zeta))d\zeta = \lambda f(0) + \int_0^Z \lambda_f \mu_f [\phi_f'(\zeta)]^{1/p} f'(\phi_f(\zeta))d\zeta
\]
\[ \lambda \left[ f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta \right] = \\
\lambda_f \left[ f(0) + \mu_f \int_0^z \left[ \phi_f'(\zeta) \right] \frac{1}{p} f'\left( \phi_f(\zeta) \right) d\zeta \right] \]

**Proof.**

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Taking the derivative of (1), we get:

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\lambda f(0) + \int_0^z \lambda F(\zeta)f'(\phi(\zeta))d\zeta = \lambda_f f(0) + \int_0^z \lambda_f \mu_f \left[ \phi_f'(\zeta) \right] \frac{1}{p} f'\left( \phi_f(\zeta) \right) d\zeta
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Then, \( \lambda_f = \lambda, \ \forall f \)
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$$\lambda [f(0) + \int_0^Z F(\zeta) f'(\phi(\zeta)) \, d\zeta] = \lambda_f [f(0) + \mu_f \int_0^Z [\phi_f'(\zeta)]^{\frac{1}{p}} f'(\phi_f(\zeta)) \, d\zeta]$$
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\lambda \left[ f(0) + \int_0^z F(\zeta) f'(\phi(\zeta)) d\zeta \right] = \\
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Now, substituting \( \lambda_f = \lambda \) in (1), we get,

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\[\Rightarrow \int_0^z F(\zeta) f'(\phi(\zeta)) d\zeta = \mu_f \int_0^z \left[ \phi_f'(\zeta) \right] \frac{1}{p} f'(\phi_f(\zeta)) d\zeta\]
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\lambda \left[ f(0) + \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta \right] = \lambda \left[ f(0) + \mu_f \int_0^z \left[ \phi_f'(\zeta) \right]^{\frac{1}{p}} f'(\phi_f(\zeta))d\zeta \right]
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\[\implies \int_0^z F(\zeta)f'(\phi(\zeta))d\zeta = \mu_f \int_0^z \left[ \phi_f'(\zeta) \right]^{\frac{1}{p}} f'(\phi_f(\zeta))d\zeta\]

This implies \( F(z)f'(\phi(z)) = \mu_f \left[ \phi_f'(z) \right]^{\frac{1}{p}} f'(\phi_f(z))\).
\[ \lambda[f(0) + \int_0^z F(\zeta) f'(\phi(\zeta)) d\zeta] = \lambda_f[f(0) + \mu_f \int_0^z [\phi_f'(\zeta)]^{\frac{1}{p}} f'(\phi_f(\zeta)) d\zeta] \]

**Proof.**

Now, substituting \( \lambda_f = \lambda \) in (1), we get,

\[ \lambda[f(0) + \int_0^z F(\zeta)f'(\phi(\zeta)) d\zeta] = \lambda[f(0) + \mu_f \int_0^z [\phi_f'(\zeta)]^{\frac{1}{p}} f'(\phi_f(\zeta))] d\zeta \]

\[ \implies \int_0^z F(\zeta)f'(\phi(\zeta)) d\zeta = \mu_f \int_0^z [\phi_f'(\zeta)]^{\frac{1}{p}} f'(\phi_f(\zeta)) d\zeta \]

by derivative

\[ \implies F(z)f'(\phi(z)) = \mu_f [\phi_f'(z)]^{\frac{1}{p}} f'(\phi_f(z)) \]

\[ \implies F(z)f'(\phi(z)) = \mu_f f'(\phi_f(z)) \left[ e^{i\theta_f} \frac{1 - |\alpha_f|^2}{(1 - \bar{\alpha_f} z)^2} \right]^{\frac{1}{p}} \]
$F(z)f'\left(\phi(z)\right) = \mu_f f'\left(\phi_f(z)\right) \left[ e^{i\theta_f} \frac{1-|\alpha_f|^2}{(1-\alpha_f z)^2} \right]^{\frac{1}{p}}$

**Proof.**

Choose, $f(z) = z$ and denote:

$$\mu_f(z) = z = \mu_1$$

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\]

So,

\[
F(z) = \mu_1 \left[ e^{i\theta_1} \frac{1 - |\alpha_1|^2}{(1 - \alpha_1 \bar{z})^2} \right]^{\frac{1}{p}}
\]

\[ (2) \]
\[ F(z) f'(\phi(z)) = \mu_f f'(\phi_f(z)) \left[ e^{i\theta_f} \frac{1-|\alpha_f|^2}{(1-\alpha_f z)^2} \right]^{\frac{1}{p}} \]

**Proof.**

Choose, \( f(z) = z \) and denote:

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So,

\[
F(z) = \mu_1 \left[ e^{i\theta_1} \frac{1-|\alpha_1|^2}{(1-\alpha_1 z)^2} \right]^{\frac{1}{p}} \tag{2}
\]

Choose, \( f(z) = \frac{z^2}{2} \) and denote:

\[
\mu_f(z) = \frac{z^2}{2} = \mu_2 \\
\alpha_f(z) = \frac{z^2}{2} = \alpha_2
\]
$$F(z)f'(\phi(z)) = \mu_f f'(\phi_f(z)) \left[ e^{i\theta_f} \frac{1 - |\alpha_f|^2}{(1 - \bar{\alpha}_f z)^2} \right]^{\frac{1}{p}}$$

Proof.

So,

$$F(z)\phi(z) = \mu_2 \phi_2(z) \left[ e^{i\theta_2} \frac{1 - |\alpha_2|^2}{(1 - \bar{\alpha}_2 z)^2} \right]^{\frac{1}{p}} \tag{3}$$

Taking the ratio of (3) and (2) we get,
\[ F(z)f'(\phi(z)) = \mu_f f'(\phi_f(z)) \left[ e^{i\theta_f} \frac{1 - |\alpha_f|^2}{(1 - \overline{\alpha_f}z)^2} \right]^{\frac{1}{p}} \]

**Proof.**

So,

\[ F(z)\phi(z) = \mu_2 \phi_2(z) \left[ e^{i\theta_2} \frac{1 - |\alpha_2|^2}{(1 - \overline{\alpha_2}z)^2} \right]^{\frac{1}{p}} \tag{3} \]

Taking the ratio of (3) and (2) we get,

\[ \phi(z) = \frac{\mu_2}{\mu_1} \phi_2(z) \left[ e^{i(\theta_2 - \theta_1)} \left( \frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2} \right) \left( \frac{1 - \alpha_1 z}{1 - \overline{\alpha_2}z} \right)^2 \right]^{\frac{1}{p}} \]
\[ F(z) f'(\phi(z)) = \mu_f f'(\phi_f(z)) \left[ e^{i\theta_f} \frac{1-|\alpha_f|^2}{(1-\bar{\alpha}_f z)^2} \right]^{\frac{1}{p}} \]

**Proof.**

So,

\[ F(z)\phi(z) = \mu_2 \phi_2(z) \left[ e^{i\theta_2} \frac{1-|\alpha_2|^2}{(1-\bar{\alpha}_2 z)^2} \right]^{\frac{1}{p}} \tag{3} \]

Taking the ratio of (3) and (2) we get,

\[
\phi(z) = \frac{\mu_2}{\mu_1} \phi_2(z) \left[ e^{i(\theta_2-\theta_1)} \left( \frac{1-|\alpha_2|^2}{1-|\alpha_1|^2} \right) \left( \frac{1-\alpha_1 z}{1-\bar{\alpha}_2 z} \right)^2 \right]^{\frac{1}{p}} \\
= \beta \left[ \left( \frac{1-|\alpha_2|^2}{1-|\alpha_1|^2} \right) \left( \frac{1-\alpha_1 z}{1-\bar{\alpha}_2 z} \right)^2 \right]^{\frac{1}{p}} \phi_2(z) \text{ where } \beta = \frac{\mu_2}{\mu_1} [e^{i(\theta_2-\theta_1)}]^{\frac{1}{p}} \tag{4} \]
\[ \phi(z) = \beta \left[ \left( \frac{1-|\alpha_2|^2}{1-|\alpha_1|^2} \right) \left( \frac{1-\alpha_1^* z}{1-\bar{\alpha}_2 z} \right)^2 \right]^{\frac{1}{p}} \phi_2(z) \]

**Proof.**

Now, for \(|z| = 1|, \)

\[ |\phi(z)| = |\beta| \left[ |\frac{1-|\alpha_2|^2}{1-|\alpha_1|^2}| \cdot |\frac{1-\alpha_1^* z}{1-\bar{\alpha}_2 z}|^2 \right]^{\frac{1}{p}} |\phi_2(z)| \]
\[ \phi(z) = \beta \left[ \left( \frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2} \right) \left( \frac{1 - \bar{\alpha}_1 z}{1 - \bar{\alpha}_2 z} \right)^2 \right]^{\frac{1}{p}} \phi_2(z) \]

**Proof.**

Now, for \(|z| = 1\),

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|\phi(z)| = |\beta| \left[ \left| \frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2} \right| \left| \frac{1 - \bar{\alpha}_1 z}{1 - \bar{\alpha}_2 z} \right|^2 \right]^{\frac{1}{p}} |\phi_2(z)|
\]

\[\Rightarrow 1 = \left[ \left| \frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2} \left( \frac{1 - \bar{\alpha}_1 z}{1 - \bar{\alpha}_2 z} \right) \right|^2 \right]^{\frac{1}{p}} \]
\[ \phi(z) = \beta \left[ \left( \frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2} \right) \left( \frac{1 - \alpha_1^* z}{1 - \alpha_2^* z} \right)^2 \right]^{\frac{1}{p}} \phi_2(z) \]

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\]

\[
\implies 1 = \left| \sqrt{\frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2}} \left( \frac{1 - \alpha_1^* z}{1 - \alpha_2^* z} \right) \right|^2 \left[ \right]^{\frac{1}{p}}
\]

\[
\implies 1 = \left| \sqrt{\frac{1 - |\alpha_2|^2}{1 - |\alpha_1|^2}} \left( \frac{1 - \alpha_1^* z}{1 - \alpha_2^* z} \right) \right|
\]
Proof.

Define a function \( \omega : \mathbb{D} \rightarrow \mathbb{D} \) by,

\[
\omega(z) := \sqrt{\frac{1-|\alpha_2|^2}{1-|\alpha_1|^2}} \left( \frac{1-\bar{\alpha}_1 z}{1-\bar{\alpha}_2 z} \right)
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Proof.

Define, a function $\omega : \mathbb{D} \to \mathbb{D}$ by,

$$\omega(z) := \sqrt{\frac{1-|\alpha_2|^2}{1-|\alpha_1|^2}} \left( \frac{1-\bar{\alpha}_1 z}{1-\bar{\alpha}_2 z} \right)$$

Note that, $\omega$ is $\mathbb{T}$ invariant.
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Note that, \( \omega \) is \( \mathbb{T} \) invariant.

**Fact:** Any holomorphic function on \( \mathbb{D} \) which is continuous on \( \overline{\mathbb{D}} \) and leaves \( \mathbb{T} \) invariant is either constant or has at least one zero in \( \mathbb{D} \).
Proof.

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Note that, \( \omega \) is \( \mathbb{T} \) invariant.

**Fact:** Any holomorphic function on \( \mathbb{D} \) which is continuous on \( \bar{\mathbb{D}} \) and leaves \( \mathbb{T} \) invariant is either constant or has at least one zero in \( \mathbb{D} \).

Since, the only zero of \( \omega \) is at \( z = \frac{1}{\bar{\alpha}_1} \notin \mathbb{D} \), \( \omega \) is constant.
Proof.

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Then from equation (4) we get

$$\phi(z) = \frac{\mu_2}{\mu_1} \phi_2(z)$$  (5)
Proof continues

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$$\phi(z) = \frac{\mu_2}{\mu_1} \phi_2(z)$$

(5)

Since $\phi_2$ is conformal then $\phi$ is also conformal.

Therefore $T$ is of the form of a surjective isometry.
Proof continues

Proof.

That implies that, \( \alpha_1 = \alpha_2 \).

Then from equation (4) we get

\[
\phi(z) = \frac{\mu_2}{\mu_1} \phi_2(z)
\]  

(5)

Since \( \phi_2 \) is conformal then \( \phi \) is also conformal.

Therefore \( T \) is of the form of a surjective isometry.

Hence \( T \) is a surjective isometry, which proves \( S^p \) is algebraic reflexive.


Thank you!
(*) Denote, \( \phi_2(z) \) be the conformal map corresponding to the choice of \( f(z) = \frac{z^2}{2} \). Then, (4) together with \( \alpha_1 = \alpha_2 \) gives,

\[
\phi(z) = \beta \phi_2(z)
\]

Then, (2) implies,

\[
F(z) = \mu_1 [\phi_1'(z)]^{\frac{1}{p}}
= \gamma' [\phi_2'(z)]^{\frac{1}{p}}
= \mu_1 \cdot (e^{i\theta_1})^{\frac{1}{p}}
= \gamma' \beta \phi'(z)^{\frac{1}{p}}
= \mu \phi'(z), \mu = \gamma' \beta
\]