Pairs of operators coinciding on the orthocomplement of the sum of kernels

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The setting and outline

- Our setting: arbitrary complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and two bounded operators between them: $A, B : \mathcal{H} \to \mathcal{K}$ having a special property:

$$A$$ and $$B$$ coincide on

$$(\text{Ker}(A) + \text{Ker}(B))\perp = \text{Ker}(A)\perp \cap \text{Ker}(B)\perp = \text{cl}(\text{Im}(A^*)) \cap \text{cl}(\text{Im}(B^*)).$$

- Particularly interesting are the pairs when the same holds for $A^*$ and $B^*$. 

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  - $A$ and $B$ coincide on $(\text{Ker}(A) + \text{Ker}(B))^\perp = \text{Ker}(A)^\perp \cap \text{Ker}(B)^\perp = \overline{\text{cl}(\text{Im}(A^*))} \cap \overline{\text{cl}(\text{Im}(B^*))}$.

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Outline

- A few examples.
- Results regarding the geometry of ranges of such pairs.
- Results on invertibility of the sum $A + B$.
- A quick application.
The setting and outline

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\[
A \text{ and } B \text{ coincide on } \quad (\ker(A) + \ker(B))^\perp = \ker(A)^\perp \cap \ker(B)^\perp = \text{cl}(\text{Im}(A^*)) \cap \text{cl}(\text{Im}(B^*)).
\]

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Outline

- A few examples.
- Results regarding the geometry of ranges of such pairs.
- Results on invertibility of the sum \( A + B \).
- A quick application.

Many results here are from a joint work with Alejandra Maestripieri.
Examples

Reminder

\[ A = B \text{ on } (\ker(A) + \ker(B))^\perp = \ker(A)^\perp \cap \ker(B)^\perp = \text{cl}(\text{Im}(A^*)) \cap \text{cl}(\text{Im}(B^*)). \]
Examples

Reminder

\[ A = B \text{ on } (\text{Ker}(A) + \text{Ker}(B))^\perp = \text{Ker}(A)^\perp \cap \text{Ker}(B)^\perp = \text{c} \ell(\text{Im}(A^*)) \cap \text{c} \ell(\text{Im}(B^*)). \]

- Pairs of orthogonal projections \( P_M, P_N \).
- More generally: pairs \( PQ \) and \( QP \).
- Pairs for which \( \text{Ker}(A)^\perp \cap \text{Ker}(B)^\perp = \{0\} \) and/or \( \text{Ker}(A^*)^\perp \cap \text{Ker}(B^*)^\perp = \{0\} \) (studied on many occasions in different forms).
- Parallel sum of operators.
Examples

Reminder

\[ A = B \text{ on } (\ker(A) + \ker(B))^\perp = \ker(A)^\perp \cap \ker(B)^\perp = c\ell(\im(A^*)) \cap c\ell(\im(B^*)). \]

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- Parallel sum of operators.

- We can say that our results extend results for these classes of operators, but this wasn’t our goal.
Range additivity and closed ranges

- Obviously: $\text{Im}(A + B) \subseteq \text{Im}(A) + \text{Im}(B)$.

Operators $A, B : \mathcal{H} \to \mathcal{K}$ are **range additive** if: $\text{Im}(A + B) = \text{Im}(A) + \text{Im}(B)$.

- Algebra + geometry.
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Algebra + geometry.

Positive operators \( A \) and \( B \) are always **almost** range additive: since \( \ker(A + B) = \ker(A) \cap \ker(B) \), we get: \( \text{cl}(\text{Im}(A + B)) = \text{cl}(\text{Im}(A) + \text{Im}(B)) \).
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  In fact: \( \text{Im}(A + B) \) is closed iff \( \text{Im}(A) + \text{Im}(B) \) is closed. (Arias et al. 2013)
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- Two orthogonal projections \( P_M \) and \( P_N \) are range additive iff \( M + N \) is closed. (Folklore; Anderson and Schreiber 1971)
Range additivity and closed ranges

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- Two orthogonal projections $P_M$ and $P_N$ are range additive iff $M + N$ is closed. (Folklore; Anderson and Schreiber 1971)

- If $\text{Ker}(A) + \text{Ker}(B) = \mathcal{H}$, then $A$ and $B$ are range additive. If moreover $\text{Im}(A) \cap \text{Im}(B) = \{0\}$, the opposite implication is also true. (Maestripieri 2014; Arias et al. 2015)
Our results on range additivity and closed ranges

Theorem

If $A = B$ on $(\ker(A) + \ker(B))^\perp$, then:

1. $A$ and $B$ are almost range additive: $c_\ell(\im(A + B)) = c_\ell(\im(A) + \im(B))$.

2. If $\im(A + B)$ is closed, then so is $\im(A) + \im(B)$, and so $\im(A + B) = \im(A) + \im(B)$.

However, if $\im(A) + \im(B)$ is closed, $\im(A + B)$ does not need to be.

Theorem

If $A = B$ on $(\ker(A) + \ker(B))^\perp$, and $A^* = B^*$ on $(\ker(A^*) + \ker(B^*))^\perp$, then:

$\im(A + B)$ is closed iff both sums $\im(A) + \im(B)$ and $\im(A^*) + \im(B^*)$ are closed.

In that case, $\im(A)$ and $\im(B)$ are closed, and $\im(A + B) = \im(A) + \im(B)$.
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If \( A = B \) on \((\text{Ker}(A) + \text{Ker}(B))\)\(^\perp\), then:

1. **A and B are almost range additive**: \( c\ell(\text{Im}(A + B)) = c\ell(\text{Im}(A) + \text{Im}(B)) \).

2. If \( \text{Im}(A + B) \) is closed, then so is \( \text{Im}(A) + \text{Im}(B) \), and so \( \text{Im}(A + B) = \text{Im}(A) + \text{Im}(B) \).

However, if \( \text{Im}(A) + \text{Im}(B) \) is closed, \( \text{Im}(A + B) \) does not need to be.
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If $A = B$ on $(\ker(A) + \ker(B))^\perp$, then:

1. $A$ and $B$ are almost range additive: $\text{cl}(\text{Im}(A + B)) = \text{cl}(\text{Im}(A) + \text{Im}(B))$.

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If $\text{Im}(PQ + QP)$ is closed, then so is $\text{Im}(PQ)$. 

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A and $B$ coincide on $(\ker(A) + \ker(B))^\perp$
Our results on range additivity and closed ranges

**Theorem**

*If \( A = B \) on \((\text{Ker}(A) + \text{Ker}(B))\)\(^\perp\), then:*

1. *A and B are almost range additive:* \( \text{cl}(\text{Im}(A + B)) = \text{cl}(\text{Im}(A) + \text{Im}(B)) \).
2. *If \( \text{Im}(A + B) \) is closed, then so is \( \text{Im}(A) + \text{Im}(B) \), and so \( \text{Im}(A + B) = \text{Im}(A) + \text{Im}(B) \).*

*However,* *if \( \text{Im}(A) + \text{Im}(B) \) is closed, \( \text{Im}(A + B) \) does not need to be.*
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Theorem

If \( A = B \) on \((\ker(A) + \ker(B))\perp\), then:

1. \textit{A and B are almost range additive:} \( c_\ell(\text{Im}(A + B)) = c_\ell(\text{Im}(A) + \text{Im}(B)) \).

2. \textit{If Im}(A + B) \text{ is closed, then so is} \text{Im}(A) + \text{Im}(B), \text{ and so } \text{Im}(A + B) = \text{Im}(A) + \text{Im}(B).

\textit{However, if Im}(A) + \text{Im}(B) \text{ is closed, Im}(A + B) \text{ does not need to be.}

Theorem

If \( A = B \) on \((\ker(A) + \ker(B))\perp\), and \( A^* = B^* \) on \((\ker(A^*) + \ker(B^*))\perp\), then:
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**Theorem**

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, then:

1. *A and B are almost range additive*: $c\ell(\text{Im}(A + B)) = c\ell(\text{Im}(A) + \text{Im}(B))$.
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If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$, then:

*Im($A + B$) is closed iff both sums $\text{Im}(A) + \text{Im}(B)$ and $\text{Im}(A^*) + \text{Im}(B^*)$ are closed.*
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In that case, $\text{Im}(A)$ and $\text{Im}(B)$ are closed, and $\text{Im}(A + B) = \text{Im}(A) + \text{Im}(B)$. 
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Theorem

If \( A = B \) on \((\text{Ker}(A) + \text{Ker}(B))\)\(\perp\), then:

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**However,** if \( \text{Im}(A) + \text{Im}(B) \) is closed, \( \text{Im}(A + B) \) does not need to be.

Theorem

If \( A = B \) on \((\text{Ker}(A) + \text{Ker}(B))\)\(\perp\), and \( A^* = B^* \) on \((\text{Ker}(A^*) + \text{Ker}(B^*))\)\(\perp\), then:

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If \( \text{Im}(PQ + QPQPQPQPQPQP) \) is closed then so is \( \text{Im}(PQ) \).
Theorem

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$, then:

$$\text{Im} (A + B) = \text{Im} (A) + \text{Im} (B)$$

iff

$$\text{Ker}(A) \perp \text{Ker}(B)$$

is closed.

$$\text{Im} (PQ + QP) = \text{Im} (PQ) + \text{Im} (QP)$$

iff

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is closed. Hence, if

$$\text{Im} (PQ) + \text{Im} (QP)$$

is closed, then

$$\text{Im} (PQ)$$

is also closed, etc.
Theorem

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$, then:

$\text{Im}(A + B) = \text{Im}(A) + \text{Im}(B)$ if and only if $\text{Ker}(A)^\perp + \text{Ker}(B)^\perp$ is closed.
Theorem

If \( A = B \) on \( (\text{Ker}(A) + \text{Ker}(B)) \perp \), and \( A^* = B^* \) on \( (\text{Ker}(A^*) + \text{Ker}(B^*)) \perp \), then:

\[ \text{Im}(A + B) = \text{Im}(A) + \text{Im}(B) \quad \text{iff} \quad \text{Ker}(A) \perp + \text{Ker}(B) \perp \text{ is closed.} \]

\[ \text{Im}(PQ + QP) = \text{Im}(PQ) + \text{Im}(QP) \quad \text{iff} \quad \text{Im}(PQ) + \text{Im}(QP) \text{ is closed.} \]

Hence, if \( \text{Im}(PQ) + \text{Im}(QP) \) is closed, then \( \text{Im}(PQ) \) is also closed, etc.
Inverting the sum \((A + B)^{-1}\)

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(Werner 1987) If \(A\) and \(B\) are square matrices such that \(\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)\) then

\[(A + B)^g = A^g + B^g \quad \text{and} \quad (A + B)^{-1} = A^g + B^g.\]
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  \[(A + B)^g = A^g + B^g\] and \((A + B)^{-1} = A^g + B^g\).

- (Du, Deng, Mbekhta, Müller 2007) If \(P\) and \(Q\) are projections on a Banach space, invertibility (and many other properties) is invariant for linear combinations \(\alpha P + \beta Q\), as long as \(\alpha, \beta \neq 0\) and \(\alpha + \beta \neq 0\) (nice pair).
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  then 
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  and 
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- (Arias, Corach, Gonzales 2013) If \(A\) and \(B\) are positive, then \(A + B\) is invertible iff \(\text{Im}(A) + \text{Im}(B) = \mathcal{H}\).
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- (Arias, Corach, Maestripieri 2015) If 
  \[
  \text{Ker}(A)^\perp \cap \text{Ker}(B)^\perp = \{0\}, \quad \text{Ker}(A^*)^\perp \cap \text{Ker}(B^*)^\perp = \{0\}
  \]
  and if \(A\) and \(B\) are closed range operators which are also range additive, then: 
  \[
  (A + B)^\dagger = PA^\dagger Q + (I - P)B^\dagger (I - Q).
  \]
Invertibility and closed ranges

**Theorem**

If \( A = B \) on \( (\text{Ker}(A) + \text{Ker}(B))^\perp \), and \( A^* = B^* \) on \( (\text{Ker}(A^*) + \text{Ker}(B^*))^\perp \), then:

\[ A + B \text{ is invertible iff } \text{Im}(A) + \text{Im}(B) = K \text{ and } \text{Im}(A^*) + \text{Im}(B^*) = H. \]

(reminder) \( A + B \) has a closed range iff \( \text{Im}(A) + \text{Im}(B) \) and \( \text{Im}(A^*) + \text{Im}(B^*) \) are closed.
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If $A = B$ on $(\ker(A) + \ker(B))^\perp$, and $A^* = B^*$ on $(\ker(A^*) + \ker(B^*))^\perp$, then:

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Invertibility and closed ranges

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A + B \text{ is invertible iff } \text{Im}(A) + \text{Im}(B) = \mathcal{K} \text{ and } \text{Im}(A^*) + \text{Im}(B^*) = \mathcal{H}.
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*(reminder) \( A + B \) has a closed range iff \( \text{Im}(A) + \text{Im}(B) \) and \( \text{Im}(A^*) + \text{Im}(B^*) \) are closed.*
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**Theorem**

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$, and $\alpha, \beta$ is a nice pair, then:

$A + B$ is invertible iff $\alpha A + \beta B$ is invertible;
Invertibility and closed ranges

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If \( A = B \) on \((\ker(A) + \ker(B))\perp\), and \( A^* = B^* \) on \((\ker(A^*) + \ker(B^*))\perp\), then:

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**Theorem**

If \( A = B \) on \((\ker(A) + \ker(B))\perp\), and \( A^* = B^* \) on \((\ker(A^*) + \ker(B^*))\perp\), and \( \alpha, \beta \) is a nice pair, then:

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\( A + B \) has a closed range iff \( \alpha A + \beta B \) has a closed range.
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If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$:
Formulas

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If $A + B$ is invertible, then:

\[
(A + B)^{-1} = A_{\text{Ker}(B), \text{Im}(B)}^{(2)} + B_{\text{Ker}(A), \text{Im}(A)}^{(2)} + \frac{1}{2} A^\dagger P(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp;
\]
Formulas

Theorem

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$:

1. If $A + B$ is invertible, then:

   $$(A + B)^{-1} = A^{(2)}_{\text{Ker}(B), \text{Im}(B)} + B^{(2)}_{\text{Ker}(A), \text{Im}(A)} + \frac{1}{2} A^\dagger P (\text{Ker}(A^*) + \text{Ker}(B^*))^\perp;$$

2. If $\text{Im}(A + B)$ is closed, then:

   $$(A + B)^G = A^g + B^g + \frac{1}{2} PA^\dagger Q.$$
Theorem

If $A = B$ on $(\text{Ker}(A) + \text{Ker}(B))^\perp$, and $A^* = B^*$ on $(\text{Ker}(A^*) + \text{Ker}(B^*))^\perp$:

1. If $A + B$ is invertible, then:

   $$(A + B)^{-1} = A_{\text{Ker}(B),\text{Im}(B)}^{(2)} + B_{\text{Ker}(A),\text{Im}(A)}^{(2)} + \frac{1}{2} A^\dagger P (\text{Ker}(A^*) + \text{Ker}(B^*))^\perp;$$

2. If $\text{Im}(A + B)$ is closed, then:

   $$(A + B)^G = A^g + B^g + \frac{1}{2} PA^\dagger Q.$$ 

The formula for the Moore-Penrose inverse $(A + B)^\dagger$ is in fact useful.
One interesting partial order on $\mathcal{B}(\mathcal{H})$

- For two orthogonal projections, $Q \leq P$ is equivalent with: $\text{Im}(P - Q) \perp \text{Im}(Q)$.
One interesting partial order on $B(\mathcal{H})$

- For two orthogonal projections, $Q \leq P$ is equivalent with:
  $\text{Im}(P - Q) \perp \text{Im}(Q)$.

- We say that $A \preceq B$ if $\text{Im}(B - A) \perp \text{Im}(A)$ and $\text{Im}(B^* - A^*) \perp \text{Im}(A^*)$. This order is called the **star partial order**.
One interesting partial order on $B(\mathcal{H})$

- For two orthogonal projections, $Q \leq P$ is equivalent with: $\text{Im}(P - Q) \perp \text{Im}(Q)$.

- We say that $A \preceq B$ if $\text{Im}(B - A) \perp \text{Im}(A)$ and $\text{Im}(B^* - A^*) \perp \text{Im}(A^*)$. This order is called the star partial order.

Theorem (Mitra 1986, Holladay)

For matrices $A$ and $B$, if $\star \sup(A, B)$ exists, then: $\star \inf(A, B) = 2A(A + B)^\dagger B$.
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For matrices $A$ and $B$, if $\sup^*(A, B)$ exists, then: $\inf^*(A, B) = 2A(A + B)^\dagger B$.

If $\sup^*(A, B)$ exists, then $A, B$ and $A^*, B^*$ are "our pairs".
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**Theorem**

For matrices the formula $\inf^*(A, B) = 2A(A + B)^\dagger B$ characterizes "our pairs".

Similarly, but not completely the same, for operators in general.
Thank you!