On isometries on some Banach spaces

— something old, something new,

something borrowed, something blue,

Part I

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Something old, something new,
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is referred to the collection of items that
helps to guarantee fertility and prosperity.
Isometries are maps between metric spaces which preserve distance between elements.

**Definition**

Let \((\mathcal{X}, |\cdot|)\) and \((\mathcal{Y}, \|\cdot\|)\) be two normed spaces over the same field. A linear map \(\varphi: \mathcal{X} \to \mathcal{Y}\) is called a *linear isometry* if

\[
\|\varphi(x)\| = |x|, \quad x \in \mathcal{X}.
\]

We shall be interested in *surjective* linear isometries on Banach spaces.

One of the main problems is to give explicit description of isometries on a particular space.
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Richard J. Fleming, James E. Jamison,


This talk is dedicated to the memory of Professor James Jamison.
Trivial isometries are isometries of the form $\lambda I$ for some $\lambda \in \mathbb{T}$, where $\mathbb{T} = \{ \lambda \in \mathbb{F} : |\lambda| = 1 \}$.

The spectrum of a surjective linear isometry is contained in $\mathbb{T}$.

For any Banach space $\mathcal{X}$ (real or complex) there is a norm $\| \cdot \|$ on $\mathcal{X}$, equivalent to the original one, such that $(\mathcal{X}, \| \cdot \|)$ has only trivial isometries (K. Jarosz, 1988).
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Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, equipped with the norm $\| \cdot \|$ induced by the inner product

$$\langle x, y \rangle = \text{tr} (xy^*) = y^*x$$

(Frobenius norm).

Then $U$ is a linear isometry on $(\mathcal{V}, \| \cdot \|)$ if and only if the following holds.

- If $\mathbb{F} = \mathbb{C}$: $U$ is a unitary operator on $\mathcal{V}$, that is,

  $$U^* U = UU^* = I.$$

- If $\mathbb{F} = \mathbb{R}$: $U$ is an orthogonal operator on $\mathcal{V}$, that is,

  $$U^t U = UU^t = I.$$
Theorem (I. Schur, 1925)

Linear isometries of $M_n(\mathbb{C})$ equipped with the spectral norm (operator norm) have one of the following forms:

$$X \mapsto UXV \quad \text{or} \quad X \mapsto UX^tV,$$

where $U, V \in M_n(\mathbb{C})$ are unitaries.
Unitarily invariant norms on $M_n(\mathbb{F})$

[C.K. Li, N.K. Tsing, 1990]

Let $G$ be the group of all linear operators of the form $X \mapsto UXV$ for some fixed unitary (orthogonal) $U, V \in M_n(\mathbb{F})$.

A norm $\| \cdot \|$ on $M_n(\mathbb{F})$ is called a **unitarily invariant norm** if $\|g(X)\| = \|X\|$ for all $g \in G, X \in M_n(\mathbb{F})$.

If $\| \cdot \|$ is a unitarily invariant norm (which is not a multiple of the Frobenius norm) on $M_n(\mathbb{F}) \neq M_4(\mathbb{R})$ then its isometry group is $\langle G, \tau \rangle$, where $\tau: M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$ is the transposition operator.
In the case of $M_4(\mathbb{R})$ the isometry group is $\langle G, \tau \rangle$ or $\langle G, \tau, \alpha \rangle$, with $\alpha: M_4(\mathbb{R}) \to M_4(\mathbb{R})$ defined by

$$\alpha(X) = (X + B_1XC_1 + B_2XC_2 + B_3XC_3)/2,$$

where

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$
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Unitary congruence invariant norms on $S_n(\mathbb{C})$

[C.K. Li, N.K. Tsing, 1990-1991]

Let $G$ be the group of all linear operators of the form $X \mapsto U^t X U$ for some fixed unitary (orthogonal) $U \in M_n(\mathbb{F})$.

A norm $\| \cdot \|$ on $V \in \{ S_n(\mathbb{C}), K_n(\mathbb{F}) \}$ is called a unitary congruence invariant norm if $\| g(X) \| = \| X \|$ for all $g \in G$, $X \in V$.

If $\| \cdot \|$ is a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, then its isometry group is $G$. 
If $\|\cdot\|$ is a unitary congruence invariant norm on $K_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, then its isometry group is $G$ if $n \neq 4$, and $\langle G, \gamma \rangle$ if $n = 4$, where $\gamma(X)$ is obtained from $X$ by interchanging its $(1,4)$ and $(2,3)$ entries, and interchanging its $(4,1)$ and $(3,2)$ entries accordingly.

If $\|\cdot\|$ is a unitary congruence invariant norm on $K_n(\mathbb{R})$, which is not a multiple of the Frobenius norm, then its isometry group is $\langle G, \tau \rangle$ if $n \neq 4$, and $\langle G, \tau, \gamma \rangle$ if $n = 4$. 

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On isometries on some Banach spaces
Let $C_0(\Omega)$ be the algebra of all continuous complex-valued functions on a locally compact Hausdorff space $\Omega$, vanishing at infinity.

**Theorem (Banach–Stone)**

Let $T : C_0(\Omega_1) \to C_0(\Omega_2)$ be a surjective linear isometry. Then there exist a homeomorphism $\varphi : \Omega_2 \to \Omega_1$ and a continuous unimodular function $u : \Omega_2 \to \mathbb{C}$ such that

$$T(f)(\omega) = u(\omega)f(\varphi(\omega)), \quad f \in C_0(\Omega_1), \ \omega \in \Omega_2.$$ 

The first (Banach’s) version of this theorem (1932): for real-valued functions on compact metric spaces. Stone (1937): for real-valued functions on compact Hausdorff spaces.
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The first (Banach’s) version of this theorem (1932): for real-valued functions on compact metric spaces.

A C*-algebra is a complex Banach *-algebra \((\mathcal{A}, \| \cdot \|)\) such that \(\| a^* a \| = \| a \|^2\) for all \(a \in \mathcal{A}\).

Example

- \(\mathbb{C}\) = complex numbers,
- \(\mathcal{B}(\mathcal{H})\) = all bounded linear operators on a complex Hilbert space \(\mathcal{H}\),
- \(\mathcal{K}(\mathcal{H})\) = all compact operators on a complex Hilbert space \(\mathcal{H}\),
- \(\mathcal{C}(\Omega)\) = all continuous complex-valued functions on a compact Hausdorff space \(\Omega\),
- \(\mathcal{C}_0(\Omega)\) = all continuous complex-valued functions on a locally compact Hausdorff space \(\Omega\), vanishing at infinity.
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Isometries of $C^*$-algebras

**Theorem (R. Kadison, 1951)**

Let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$-algebras and $T: \mathcal{A} \to \mathcal{B}$ a surjective linear isometry. Then $T = UJ$, where $J: \mathcal{A} \to \mathcal{B}$ is a Jordan $\ast$-isomorphism (that is, a linear map satisfying $J(a^2) = J(a)^2$ and $J(a^\ast) = J(a)^\ast$ for every $a \in \mathcal{A}$) and a unitary element $U \in \mathcal{B}$.

**Theorem (A. Paterson, A. Sinclair, 1972)**

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and $T: \mathcal{A} \to \mathcal{B}$ a surjective linear isometry. Then $T = UJ$, where $J: \mathcal{A} \to \mathcal{B}$ is a Jordan $\ast$-isomorphism, and $U$ on $\mathcal{B}$ is unitary such that there exists $V$ on $\mathcal{B}$ satisfying $aU(b) = V(a)b$ for all $a, b \in \mathcal{B}$. 
**Theorem (R. Kadison, 1951)**

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Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $T: \mathcal{A} \to \mathcal{B}$ a surjective linear isometry. Then $T = UJ$, where $J: \mathcal{A} \to \mathcal{B}$ is a Jordan $\ast$-isomorphism, and $U$ on $\mathcal{B}$ is unitary such that there exists $V$ on $\mathcal{B}$ satisfying $aU(b) = V(a)b$ for all $a, b \in \mathcal{B}$. 
Isometries of $B(\mathcal{H})$

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Throughout we fix an orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}$.

Let $T \in B(\mathcal{H})$. If $S \in B(\mathcal{H})$ is such that $\langle Te_\lambda, e_\mu \rangle = \langle Se_\mu, e_\lambda \rangle$ for all $\lambda, \mu \in \Lambda$, then $S$ is called the transpose of $T$ associated to the basis $\{e_\lambda : \lambda \in \Lambda\}$ and it is denoted by $T^t$.

**Theorem**

Let $T : B(\mathcal{H}) \to B(\mathcal{H})$ be a surjective linear isometry. Then there exist unitary $U, V \in B(\mathcal{H})$ such that $T$ has one of the following forms:

$$X \mapsto UXV \quad \text{or} \quad X \mapsto UX^tV.$$
A **JB***-**triple** is a complex Banach space $\mathcal{A}$ together with a continuous triple product $\{xyz\} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

(i) $\{xyz\}$ is linear in $x$ and $z$ and conjugate linear in $y$;
(ii) $\{xyz\} = \{zyx\}$;
(iii) for any $x \in \mathcal{A}$, the operator $\delta(x) : \mathcal{A} \to \mathcal{A}$ defined by $\delta(x)y = \{xxy\}$ is hermitian with nonnegative spectrum;
(iv) $\delta(x)\{abc\} = \{\delta(x)a, b, c\} − \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}$;
(v) for every $x \in \mathcal{A}$, $\|\{xxx\}\| = \|x\|^3$.

**Example**

- complex Hilbert spaces: $\{xyz\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x)$
- C*-algebras, $S(\mathcal{H}), A(\mathcal{H})$: $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$, where $S(\mathcal{H}) = \{T \in B(\mathcal{H}) : T^t = T\}$ symmetric operators, $A(\mathcal{H}) = \{T \in B(\mathcal{H}) : T^t = −T\}$ antisymmetric operators.
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Isometries on JB*-triples

Theorem (W. Kaup, 1983)

Let $\mathcal{A}$ be a JB*-triple. Then every surjective linear isometry $T: \mathcal{A} \to \mathcal{A}$ satisfies

$$T(\{xyz\}) = \{T(x)T(y)T(z)\}, \quad x, y, z \in \mathcal{A}.$$

In particular, if $\mathcal{A}$ is a C*-algebra then

$$T(xy^*x) = T(x)T(y)^*T(x), \quad x, y \in \mathcal{A}.$$
Theorem (W. Kaup, 1983)

Let $A$ be a JB*-triple. Then every surjective linear isometry $T : A \to A$ satisfies

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Isometries on $S(H)$ and $A(H)$

Every surjective linear isometry $T : A \rightarrow A$, where $A$ is $S(H)$ or $A(H)$, satisfies

$$T(XY^*X) = T(X)T(Y)^*T(X)$$

for all $X, Y \in A$.

The following theorem gives an explicit formula for $T$.

**Theorem (A. Fošner and D. I., 2011)**

Let $A$ be $S(H)$ or $A(H)$ and let $T : A \rightarrow A$ be a surjective linear isometry. Then there exists a unitary $U \in B(H)$ such that $T$ has the form $X \mapsto UXU^t$. 

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On isometries on some Banach spaces
Isometries on $S(\mathcal{H})$ and $A(\mathcal{H})$

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The following theorem gives an explicit formula for $T$.

**Theorem (A. Fošner and D. I., 2011)**

Let $\mathcal{A}$ be $S(\mathcal{H})$ or $A(\mathcal{H})$ and let $T : \mathcal{A} \rightarrow \mathcal{A}$ be a surjective linear isometry. Then there exists a unitary $U \in B(\mathcal{H})$ such that $T$ has the form $X \mapsto UXU^t$. 
A minimal norm ideal \((\mathcal{I}, \nu)\) consists of a two-sided proper ideal \(\mathcal{I}\) in \(B(\mathcal{H})\) together with a norm \(\nu\) on \(\mathcal{I}\) satisfying the following:

- the set of all finite rank operators on \(\mathcal{H}\) is dense in \(\mathcal{I}\),
- \(\nu(X) = \|X\|\) for every rank one operator \(X\),
- \(\nu(UXV) = \nu(X)\) for every \(X \in \mathcal{I}\) and all unitary \(U, V \in B(\mathcal{H})\).

**Theorem (A. Sourour, 1981)**

If \(\mathcal{I}\) is different from the Hilbert-Schmidt class then every surjective linear isometry on \(\mathcal{I}\) has the form \(X \mapsto UXV\) or \(X \mapsto UX^tV\) for some unitary \(U, V \in B(\mathcal{H})\).
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**Hermitian operators**

**Definition**

Let $\mathcal{X}$ be a complex Banach space. A bounded linear operator $T: \mathcal{X} \rightarrow \mathcal{X}$ is said to be hermitian if $e^{i\varphi T}$ is an isometry for all $\varphi \in \mathbb{R}$.

**Example**

$C^1[0,1]$, the space of continuously differentiable complex-valued functions on $[0,1]$ with $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$, admits only trivial hermitian operators, that is, real multiples of $I$ (E. Berkson, A. Sourour, 1974).

**Example**

Hermitian operators on a C*-algebra $A$ have the form $x \mapsto ax + xb$ for some self-adjoint $a, b \in M(A)$. 
**Definition**

Let $\mathcal{X}$ be a complex Banach space. A bounded linear operator $T : \mathcal{X} \rightarrow \mathcal{X}$ is said to be **hermitian** if $e^{i\varphi}T$ is an isometry for all $\varphi \in \mathbb{R}$.

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**Example**

Hermitian operators on a $C^*$-algebra $A$ have the form $x \mapsto ax + xb$ for some self-adjoint $a, b \in M(A)$. 
By a **projection** on a complex Banach space we mean a linear operator $P$ such that $P^2 = P$.

**Theorem (J. Jamison, 2007)**

A projection $P$ on a complex Banach space is a hermitian projection if and only if $P + \lambda(I - P)$ is an isometry for all $\lambda \in \mathbb{T}$, where $\mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.

**Example**

$C^1[0, 1]$ admits only trivial hermitian projections (0 and $I$).

**Example**

Every orthogonal projection on a complex Hilbert space is hermitian.
By a projection on a complex Banach space we mean a linear operator $P$ such that $P^2 = P$.

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**Example**

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**Example**

Every orthogonal projection on a complex Hilbert space is hermitian.
Theorem (L.L. Stachó and B. Zalar, 2004)

(i) Let $P : B(\mathcal{H}) \to B(\mathcal{H})$ be a hermitian projection. Then $P$ has the form $X \mapsto QX$ or $X \mapsto XQ$ for some $Q \in B(\mathcal{H})$ such that $Q = Q^* = Q^2$.

(ii) Let $P : S(\mathcal{H}) \to S(\mathcal{H})$ be a hermitian projection. Then either $P = 0$ or $P = I$.

(iii) Let $P : A(\mathcal{H}) \to A(\mathcal{H})$ be a hermitian projection. Then $P$ or $I - P$ has the form $X \mapsto QX + XQ^t$ with $Q = x \otimes x$ for some unit vector $x \in \mathcal{H}$.
Hermitian projections on C*-algebras

**Theorem (M. Fošner and D. I., 2005)**

Let $\mathcal{A}$ be a C*-algebra and let $P: \mathcal{A} \to \mathcal{A}$ be a hermitian projection. Then there exist a $\ast$-ideal $\mathcal{I}$ of $\mathcal{A}$ and $p = p^* = p^2 \in M(\mathcal{I}^\perp \oplus \mathcal{I}^\perp\perp)$ such that $P(x) = px$ for all $x \in \mathcal{I}^\perp$ and $P(x) = xp$ for all $x \in \mathcal{I}^\perp\perp$.

**Corollary**

Let $\Omega$ be a locally compact Hausdorff space. Then $P: C_0(\Omega) \to C_0(\Omega)$ is a hermitian projection if and only if $Pf = 1_Yf$, where $1_Yf$ is the indicator function on a proper component $Y$ of $\Omega$.

In particular, if $\Omega$ is connected then $C_0(\Omega)$ admits only trivial hermitian projections.
Corollary

Let $A$ be $K(H)$ or $B(H)$ and let $P: A \to A$ be a hermitian projection. Then there exists $p = p^* = p^2 \in B(H)$ such that $P$ has the form $x \mapsto px$ or $x \mapsto xp$.

Theorem (J. Jamison, 2007)

Let $I$ be a minimal norm ideal in $B(H)$, different from the Hilbert-Schmidt class, and let $P: I \to I$ be a hermitian projection. Then $P$ has the form $X \mapsto QX$ or $X \mapsto XQ$ for some $Q = Q^* = Q^2 \in B(H)$. 
Corollary

Let $\mathcal{A}$ be $K(\mathcal{H})$ or $B(\mathcal{H})$ and let $P : \mathcal{A} \to \mathcal{A}$ be a hermitian projection. Then there exists $p = p^* = p^2 \in B(\mathcal{H})$ such that $P$ has the form $x \mapsto px$ or $x \mapsto xp$.

Theorem (J. Jamison, 2007)

Let $\mathcal{I}$ be a minimal norm ideal in $B(\mathcal{H})$, different from the Hilbert-Schmidt class, and let $P : \mathcal{I} \to \mathcal{I}$ be a hermitian projection. Then $P$ has the form $X \mapsto QX$ or $X \mapsto XQ$ for some $Q = Q^* = Q^2 \in B(\mathcal{H})$. 
A generalization of hermitian projections

Recall that a projection $P$ on $\mathcal{X}$ is a hermitian projection if and only if the map

$$P + \lambda(I - P)$$

is an isometry for all $\lambda \in \mathbb{T}$.

These projections are also known as **bicircular projections**.

We can also study projections $P$ such that

$$P + \lambda(I - P)$$

is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$.

These projections are also known as **generalized bicircular projections (GBPs)**.
Generalized bicircular projections on $S_n(\mathbb{C})$

Let $\mathcal{A}$ be $S_n(\mathbb{C})$ or $K_n(\mathbb{C})$. A norm $\| \cdot \|$ on $\mathcal{A}$ is said to be a unitary congruence invariant norm if

$$\|UXU^t\| = \|X\|$$

for all unitary $U \in M_n(\mathbb{C})$ and all $X \in \mathcal{A}$.

**Theorem (M. Fošner, D. I. and C.K. Li, 2007)**

Let $\| \cdot \|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm. Suppose $P: S_n(\mathbb{C}) \to S_n(\mathbb{C})$ is a nontrivial projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P)$ is an isometry of $(S_n(\mathbb{C}), \| \cdot \|)$ if and only if $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in M_n(\mathbb{C})$ such that $P$ or $I - P$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$. 
Generalized bicircular projections on $S_n(\mathbb{C})$

Let $\mathcal{A}$ be $S_n(\mathbb{C})$ or $K_n(\mathbb{C})$. A norm $\| \cdot \|$ on $\mathcal{A}$ is said to be a unitary congruence invariant norm if

$$\| UXU^t \| = \| X \|$$

for all unitary $U \in M_n(\mathbb{C})$ and all $X \in \mathcal{A}$.

**Theorem (M. Fošner, D. I. and C.K. Li, 2007)**

Let $\| \cdot \|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm. Suppose $P : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ is a nontrivial projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P)$ is an isometry of $(S_n(\mathbb{C}), \| \cdot \|)$ if and only if $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in M_n(\mathbb{C})$ such that $P$ or $I - P$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$. 
Theorem (M. Fošner, D. I. and C.K. Li, 2007)

Let $\| \cdot \|$ be a unitary congruence invariant norm on $K_n(\mathbb{C})$, which is not a multiple of the Frobenius norm. Suppose $P : K_n(\mathbb{C}) \rightarrow K_n(\mathbb{C})$ is a nontrivial projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P)$ is an isometry of $(K_n(\mathbb{C}), \| \cdot \|)$ if and only if one of the following holds.

(i) There exists $Q = vv^*$ for a unit vector $v \in \mathbb{C}^n$ such that $P$ or $I - P$ has the form $X \mapsto QX + XQ^t$.

(ii) $\lambda = -1$, $\mathcal{K} = G$ and there exists $Q = Q^* = Q^2 \in M_n(\mathbb{C})$ such that $P$ or $I - P$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.

(iii) $(\lambda, n) = (-1, 4)$, $\psi \in \mathcal{K}$, and there is a unitary $U \in M_4(\mathbb{C})$, satisfying $\psi(U^tXU) = \overline{U}\psi(X)U^*$ for all $X \in K_4(\mathbb{C})$, such that $P$ or $I - P$ has the form $X \mapsto (X + \psi(U^tXU))/2 = (X + \overline{U}\psi(X)U^*)/2$. 

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Theorem (F. Botelho and J. Jamison, 2008)

Let $I$ be a minimal norm ideal in $B(\mathcal{H})$, different from the Hilbert-Schmidt class, and let $P : A \rightarrow A$ be a projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if one of the following holds:

(i) $P$ has the form $X \mapsto QX$ or $X \mapsto XQ$ for some $Q = Q^* = Q^2 \in B(\mathcal{H})$,

(ii) $\lambda = -1$ and $P$ has one of the following forms:

- $X \mapsto \frac{1}{2}(X + UXV)$ for some unitary $U, V \in B(\mathcal{H})$ such that $U^2 = \mu I, V^2 = \overline{\mu} I$ for some $\mu \in \mathbb{C}, |\mu| = 1$,

- $X \mapsto \frac{1}{2}(X + UX^t V)$ for some unitary $U, V \in B(\mathcal{H})$ such that $V = \pm(U^t)^*$.
Theorem (P.-K. Lin, 2008)

Let $\mathcal{X}$ be a complex Banach space and let $P: \mathcal{X} \to \mathcal{X}$ be a projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if one of the following holds:

(i) $P$ is hermitian,

(ii) $\lambda = e^{\frac{2\pi i}{n}}$ for some integer $n \geq 2$.

Furthermore, if $n$ is any integer such that $n \geq 2$, then for $\lambda = e^{\frac{2\pi i}{n}}$ there is a complex Banach space $\mathcal{X}$ and a nontrivial projection $P$ on $\mathcal{X}$ such that $P + \lambda(I - P)$ is an isometry.
**Theorem (D. I., 2010)**

Let $\mathcal{A}$ be a JB*-triple and let $P: \mathcal{A} \to \mathcal{A}$ be a projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if one of the following holds:

(i) $P$ is hermitian,

(ii) $\lambda = -1$ and $P = \frac{1}{2}(I + T)$ for some linear isometry $T: \mathcal{A} \to \mathcal{A}$ satisfying $T^2 = I$. 
Corollary

Let $\mathcal{A} = B(\mathcal{H})$ or $\mathcal{A} = K(\mathcal{H})$, and let $P: \mathcal{A} \to \mathcal{A}$ be a nonhermitian projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if $\lambda = -1$ and $P$ has one of the following forms:

- $X \mapsto \frac{1}{2}(X + UXV)$ for unitary $U, V \in B(\mathcal{H})$ such that $U^2 = \mu I, V^2 = \bar{\mu} I$ for some $\mu \in \mathbb{C}, |\mu| = 1$,

- $X \mapsto \frac{1}{2}(X + UX^t V)$ for unitary $U, V \in B(\mathcal{H})$ such that $V = \pm (U^t)^*$.
Theorem (F. Botelho, 2008)

Let $\Omega$ be a connected compact Hausdorff space and let $P : C(\Omega) \to C(\Omega)$ be a nontrivial projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if $\lambda = -1$ and there exist a homeomorphism $\varphi : \Omega \to \Omega$ satisfying $\varphi^2 = I$ and a continuous unimodular function $u : \Omega \to \mathbb{C}$ satisfying $u(\varphi(\omega)) = \overline{u(\omega)}$ for every $\omega \in \Omega$, such that

$$P(f)(\omega) = \frac{1}{2} \left( f(\omega) + u(\omega)f(\varphi(\omega)) \right), \quad f \in C_0(\Omega), \, \omega \in \Omega.$$
Theorem (D. I., 2010)

Let $\Omega$ be a locally compact Hausdorff space and let $P : C_0(\Omega) \to C_0(\Omega)$ be a projection. Then $P + \lambda(I - P)$ is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$ if and only if one of the following holds.

(i) $P$ is hermitian,

(ii) $\lambda = -1$ and there exist a homeomorphism $\varphi : \Omega \to \Omega$ satisfying $\varphi^2 = I$ and a continuous unimodular function $u : \Omega \to \mathbb{C}$ satisfying $u(\varphi(\omega)) = \overline{u(\omega)}$ for every $\omega \in \Omega$, such that

$$P(f)(\omega) = \frac{1}{2} \left( f(\omega) + u(\omega)f(\varphi(\omega)) \right), \quad f \in C_0(\Omega), \ \omega \in \Omega.$$
Corollary (A. Fošner and D. I., 2011)

Let $P : S(\mathcal{H}) \to S(\mathcal{H})$ be a nontrivial projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P)$ is an isometry if and only if $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in B(\mathcal{H})$ such that $P$ or $I - P$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.

Corollary (A. Fošner and D. I., 2011)

Let $P : A(\mathcal{H}) \to A(\mathcal{H})$ be a nontrivial projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P)$ is an isometry if and only if one of the following holds:

(i) $P$ or $I - P$ has the form $X \mapsto QX + XQ^t$, where $Q = x \otimes x$ for some norm one $x \in \mathcal{H}$,

(ii) $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in B(\mathcal{H})$ such that $P$ or $I - P$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$. 

Generalized bicircular projections
and the spectrum of the corresponding isometry

If $P$ is a projection such that

$$T \overset{\text{def}}{=} P + \lambda (I - P)$$

is an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$, then $T$ is a surjective isometry and $\sigma(T) = \{1, \lambda\}$.

Conversely, if $T$ is a surjective isometry with $\sigma(T) = \{1, \lambda\}$, $\lambda \neq 1$, then $|\lambda| = 1$ and

$$P \overset{\text{def}}{=} \frac{T - \lambda I}{1 - \lambda}$$

is a projection.
If $P$ is a projection such that

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Conversely, if $T$ is a surjective isometry with $\sigma(T) = \{1, \lambda\}$, $\lambda \neq 1$, then $|\lambda| = 1$ and

$$P \overset{\text{def}}{=} \frac{T - \lambda I}{1 - \lambda}$$

is a projection.
Every isolated point in the spectrum $\sigma(T)$ of a surjective isometry $T$ on a Banach space is an eigenvalue of $T$ with a complemented eigenspace. In particular, if $\sigma(T) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ then all $\lambda_i$'s are eigenvalues, and the associated eigenprojections $P_i$'s satisfy

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I \quad \text{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$ 

Here, we write $P \oplus Q$ to indicate that the Banach space projections $P$ and $Q$ disjoint from each other, i.e., $PQ = QP = 0$. 
**Definition**

Let $P_0$ be a nonzero projection on a Banach space $\mathcal{X}$, and $n \geq 2$. We call $P_0$ a **generalized $n$-circular projection** if there exists a (surjective) isometry $T : \mathcal{X} \to \mathcal{X}$ with $\sigma(T) = \{1, \lambda_1, \ldots, \lambda_{n-1}\}$ consisting of $n$ distinct (modulus one) eigenvalues such that $P_0$ is the eigenprojection of $T$ associated to $\lambda_0 = 1$.

In this case, there are nonzero projections $P_1, \ldots, P_{n-1}$ on $\mathcal{X}$ such that

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I \quad \text{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

We also say that $P_0$ is a generalized $n$-circular projection associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. 
Generalized $n$-circular projections on $C_0(\Omega)$

Let $\Omega$ be a locally compact Hausdorff space. Let $\varphi: \Omega \to \Omega$ be a homeomorphism with period $m$, i.e., $\varphi^m = id_{\Omega}$ and $\varphi^k \neq id_{\Omega}$ for $k = 1, 2, \ldots, m - 1$.

Let $u$ be a continuous unimodular scalar function on $\Omega$ such that

$$u(\omega) \cdots u(\varphi^{m-1}(\omega)) = 1, \quad \omega \in \Omega.$$

Then the surjective isometry $T: C_0(\Omega) \to C_0(\Omega)$ defined by

$$Tf(\omega) = u(\omega)f(\varphi(\omega))$$

satisfies $T^m = I$.

Therefore, the spectrum $\sigma(T) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ consists of $n$ distinct $m$th roots of unity.

Replacing $T$ with $\lambda_0 T$, we can assume that $\lambda_0 = 1$. 
Generalized $n$-circular projections on $C_0(\Omega)$

This gives rise to a spectral decomposition

$$I = P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1}, \quad T = \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$ 

Here, the spectral projections are defined by

$$P_i f(w) = \frac{(I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i}^{m-1} T^{m-1})f(\omega)}{m}$$

$$= \frac{1}{m} \left( f(\omega) + \overline{\lambda_i} u(\omega) f(\varphi(\omega)) + \cdots \right.$$

$$+ \overline{\lambda_i}^{m-1} u(\omega) \ldots u(\varphi^{m-2}(\omega)) f(\varphi^{m-1}(\omega)) \left) \right.$$ 

for all $f \in C_0(\Omega)$, $\omega \in \Omega$, and $i = 0, 1, \ldots, n - 1$.

An $m$th root $\lambda$ of unity does not belong to $\sigma(T)$ if and only if

$$I + \overline{\lambda} T + \cdots + \overline{\lambda}^{m-1} T^{m-1} = 0.$$
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Generalized $n$-circular projections on $C_0(\Omega)$

Theorem (D. I., C.-N. Liu and N.-C. Wong)

Let $\Omega$ be a connected locally compact space. Let $T$ be a surjective isometry of $C_0(\Omega)$ with finite spectrum consisting of $n$ points. Then all eigenvalues of $T$ are of finite orders.

Definition

We call the generalized $n$-circular projection $P_0$ periodic (resp. primitive) if it is an eigenprojection of a periodic surjective isometry $T$ of period $m \geq n$ (resp. of period $m = n$).
Example

Let $n$ be a positive integer and let $\tau = e^{i \frac{2\pi}{n}}$.
Let $\mathbb{T}$ be the unit circle in the complex plane.
Then $Tf(z) = f(\tau z)$ is a surjective isometry of $C(\mathbb{T})$, and

$$\sigma(T) = \{1, \tau, \ldots, \tau^{n-1}\}.$$
Theorem

Let $\Omega$ be a connected compact Hausdorff space and let $P_0 : C_0(\Omega) \to C_0(\Omega)$ be a projection. Then the following holds.

(i) [F. Botelho, 2008] If $T = P_0 + \lambda_1 P_1$, with $P_0 \oplus P_1 = I$, is an isometry for some $\lambda_1 \in \mathbb{T} \setminus \{1\}$ then $\sigma(T) = \{1, -1\}$.

(ii) [A. B. Abubaker and S. Dutta, 2011] If $T = P_0 + \lambda_1 P_1 + \lambda_2 P_2$, with $P_0 \oplus P_1 \oplus P_2 = I$, is an isometry for some distinct $\lambda_1, \lambda_2 \in \mathbb{T} \setminus \{1\}$ then $\sigma(T) = \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$. 
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**Generalized 4-circular projections on** \( C_0(\Omega) \) – an example

**Example**

\[
A = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in [0, 1]\},
\]
\[
B = \{(s, -s, 0) \in \mathbb{R}^3 : s \in [-1, 1]\}, \quad \Omega = A \cup B.
\]
\[
\varphi(x, y, z) = \begin{cases} 
(y, z, x), & \text{if } (x, y, z) \in A; \\
(-x, -y, -z), & \text{if } (x, y, z) \in B.
\end{cases}
\]

The isometry \( Tf \overset{\text{def}}{=} f \circ \varphi \) of period 6 has 4 eigenvalues

\[
\lambda_0 = 1, \quad \lambda_1 = -1, \quad \lambda_2 = \beta, \quad \lambda_3 = \beta^2,
\]
where \( \beta = e^{i \frac{2\pi}{3}} \).

Hence \( T = P_0 - P_1 + \beta P_2 + \beta^2 P_3 \).
Let $\Omega$ be a connected locally compact Hausdorff space. Let $\varphi : \Omega \to \Omega$ be a homeomorphism and $u$ be a unimodular continuous scalar function defined on $\Omega$. Let $P_0$ be a generalized $n$-circular projection on $C_0(\Omega)$ associated to $Tf = u \cdot f \circ \varphi$ with the spectral decomposition

$$I = P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1},$$

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

Assume all eigenvalues $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{n-1}$ of $T$ have a (minimum) finite common period $m \geq n$. In particular, all of them are $m$th roots of unity, and $T^m = I$. Then the following holds.
Theorem (continuation)

- The homeomorphism $\varphi$ has (minimum) period $m$.
- The cardinality $k(\omega)$ of the orbit $\{\omega, \varphi(\omega), \varphi^2(\omega), \ldots\}$ of each point $\omega$ under $\varphi$ is not greater than $n$.
- $m$ is the least common multiple of $k(\omega)$ for all $\omega$ in $\Omega$. 
The spectrum $\sigma(T)$ of $T$ can be written as a union of the complete set of $k(\omega)$th roots of the modulus one scalar $\alpha_\omega = u(\omega)u(\varphi(\omega)) \cdots u(\varphi^{k(\omega)-1}(\omega))$. More precisely,

$$\sigma(T) = \bigcup_{\omega \in \Omega} \{\lambda_\omega, \lambda_\omega \eta_\omega, \lambda_\omega \eta_\omega^2, \ldots, \lambda_\omega \eta_\omega^{k(\omega)-1}\},$$

where $\lambda_\omega$ and $\eta_\omega$ are primitive $k(\omega)$th roots of $\alpha_\omega$ and unity, respectively. We call the set in the union a complete cycle of $k(\omega)$th roots of unity shifted by $\lambda_\omega$. 
Theorem (continuation)

- If \( u(\omega) = 1 \) on \( \Omega \) then we can choose all \( \lambda_\omega = 1 \), and thus \( \sigma(T) \) consists of all \( k(\omega) \)th roots of unity.
- If \( m \) is a prime integer, then \( n = m \) and \( \sigma(T) \) consists of the complete cycle of \( n \)th roots of unity.
Corollary

Let $\Omega$ be a connected locally compact Hausdorff space. Then every generalized bicircular or tricircular projection $P_0$ on $C_0(\Omega)$ is primitive. In other words, $P_0$ can only be an eigenprojection of a surjective isometry $T$ on $C_0(\Omega)$ with a spectral decomposition

$$T = P_0 - (I - P_0) \quad \text{for the bicircular case,}$$

$$T = P_0 + \beta P_1 + \beta^2 P_2 \quad \text{for the tricircular case,}$$

where $\beta = e^{i\frac{2\pi}{3}}$. 
Corollary

Let \( \Omega \) be a connected locally compact Hausdorff space. Let \( Tf = u \cdot f \circ \varphi \) be a surjective isometry on \( C_0(\Omega) \) with the spectral decomposition

\[
T = P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3.
\]

Then \( \sigma(T) = \{1, \lambda_1, \lambda_2, \lambda_3\} \) can only be one of the following:

\[
\begin{align*}
&\{1, -1, i, -i\}, & \{1, -1, \beta, \beta^2\}, & \{1, -1, -\beta, -\beta^2\}, \\
&\{1, -\beta, \beta, \beta^2\}, & \{1, \beta, \beta^2, -\beta^2\}.
\end{align*}
\]

All above cases can happen. Here \( \beta = e^{i \frac{2\pi}{3}} \).
**Corollary**

Let $\Omega$ be a connected locally compact Hausdorff space. Let $Tf = u \cdot f \circ \varphi$ be a surjective isometry on $C_0(\Omega)$ with the spectral decomposition

$$T = P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4.$$  

Then $\sigma(T) = \{1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ can only be one of the following:

- $\{1, \delta, \delta^2, \delta^3, \delta^4\}$,
- $\{1, -1, \beta, -\beta, \beta^2\}$,
- $\{1, -1, \beta, -\beta, -\beta^2\}$,
- $\{1, -1, \beta^2, -\beta^2\}$,
- $\{1, \beta, -\beta, \beta^2, -\beta^2\}$.

All above cases can happen. Here, $\beta = e^{i \frac{2\pi}{3}}$ and $\delta = e^{i \frac{2\pi}{5}}$. If $u$ is a constant function, then only the first case is allowed.
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Generalized 5-circular projections on $C_0(\Omega)$ – an example

Example

\begin{align*}
A_1 &= \{(1, 0, \rho) \in \mathbb{R}^3 : \rho \in [0, \pi]\}, \\
A_2 &= \{(1, \frac{2\pi}{3}, \rho) \in \mathbb{R}^3 : \rho \in [0, \pi]\}, \\
A_3 &= \{(1, \frac{4\pi}{3}, \rho) \in \mathbb{R}^3 : \rho \in [0, \pi]\}, \\
B &= \{(r, 0, 0) \in \mathbb{R}^3 : r \in [1/2, 3/2]\}, \\
C &= \{(r, 0, \pi) \in \mathbb{R}^3 : r \in [1/2, 3/2]\}, \\
\Omega &= A_1 \cup A_2 \cup A_3 \cup B \cup C.
\end{align*}
Example

\[ \varphi(r, \theta, \rho) = \begin{cases} 
(r, \theta + \frac{2\pi}{3}, \rho), & \text{if } (r, \theta, \rho) \in A_1 \cup A_2 \cup A_3; \\
(2 - r, \theta, \rho), & \text{if } (r, \theta, \rho) \in B \cup C.
\end{cases} \]

\[ u(r, \theta, \rho) = \begin{cases} 
e^{i\frac{2\rho}{3}}, & \text{if } (r, \theta, \rho) \in A_1 \cup A_2; \\
e^{-i\frac{4\rho}{3}}, & \text{if } (r, \theta, \rho) \in A_3; \\
1, & \text{if } (r, \theta, \rho) \in B; \\
e^{i\frac{2\pi}{3}}, & \text{if } (r, \theta, \rho) \in C.
\end{cases} \]

Then \( Tf \overset{\text{def}}{=} u \cdot f \circ \varphi \) has period 6.
Example

\[
\sigma(T) = \{1, \beta, \beta^2\} \cup \{1, -1\} \cup \{\beta, -\beta\} = \{1, -1, \beta, -\beta, \beta^2\}.
\]
Non-primitive generalized $n$-circular projections on $C_0(\Omega)$

Theorem (D. I., C.-N. Liu and N.-C. Wong)

There exists a non-primitive generalized $n$-circular projection on continuous functions on a connected compact Hausdorff space for each $n \geq 4$. 
Theorem (D. I., 2017)

Let $A$ be a JB*-triple, and $P_0 : A \to A$ be a generalized $n$-circular projection, $n \geq 2$, associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. Let $\lambda_0 = 1$. Then one of the following holds.

(i) There exist $i, j, k \in \{0, 1, \ldots, n-1\}$, $k \neq i$, $k \neq j$, such that $\lambda_i \lambda_j \lambda_k \in \{\lambda_m : m = 0, 1, \ldots, n-1\}$.

(ii) All $P_0$, $P_1$, $\ldots$, $P_{n-1}$ are hermitian.

When $n = 2$: if $P$ is not hermitian then $\lambda^2 \in \{1, \lambda\}$, or $\bar{\lambda} \in \{1, \lambda\}$; hence $\lambda = -1$.

When $n = 3$: if $P$, $Q$, $R$ are not hermitian then $\lambda_1 \lambda_2 = 1$, or $\lambda_1^2 = \lambda_2$, or $\lambda_2^2 = \lambda_1$. 
Theorem (D. I., 2017)

Let $A$ be a JB*-triple, and $P_0 : A \to A$ be a generalized $n$-circular projection, $n \geq 2$, associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. Let $\lambda_0 = 1$. Then one of the following holds.

(i) There exist $i, j, k \in \{0, 1, \ldots, n-1\}$, $k \neq i$, $k \neq j$, such that $\lambda_i \lambda_j \lambda_k \in \{\lambda_m : m = 0, 1, \ldots, n-1\}$.

(ii) All $P_0, P_1, \ldots, P_{n-1}$ are hermitian.

When $n = 2$: if $P$ is not hermitian then $\lambda^2 \in \{1, \lambda\}$, or $\overline{\lambda} \in \{1, \lambda\}$; hence $\lambda = -1$.

When $n = 3$: if $P, Q, R$ are not hermitian then $\lambda_1 \lambda_2 = 1$, or $\lambda_1^2 = \lambda_2$, or $\lambda_2^2 = \lambda_1$. 
Theorem (D. I., 2017)

Let $A$ be a JB*-triple, and $P_0 : A \to A$ be a generalized $n$-circular projection, $n \geq 2$, associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. Let $\lambda_0 = 1$. Then one of the following holds.

(i) There exist $i, j, k \in \{0, 1, \ldots, n-1\}$, $k \neq i$, $k \neq j$, such that $\lambda_i \lambda_j \lambda_k \in \{\lambda_m : m = 0, 1, \ldots, n-1\}$.

(ii) All $P_0, P_1, \ldots, P_{n-1}$ are hermitian.

When $n = 2$: if $P$ is not hermitian then $\lambda^2 \in \{1, \lambda\}$, or $\overline{\lambda} \in \{1, \lambda\}$; hence $\lambda = -1$.

When $n = 3$: if $P, Q, R$ are not hermitian then $\lambda_1 \lambda_2 = 1$, or $\lambda_2^2 = \lambda_2$, or $\lambda_1^2 = \lambda_1$. 