Approximations by Discrete Operators

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We are trying to improve the convergence of sequences of functions or numbers.
Little History

- We are trying to improve the convergence of sequences of functions or numbers.
- For sequences of functions we have pointwise convergence, uniform convergence, convergence a.e., convergence in measure etc...

For sequences of numbers, Steinhaus and Fast defined "statistical convergence" in 1951. Freedman and Sember defined "A-statistical convergence" in 1981.

In 1950, Korovkin gave his approximation results for positive linear operators.

Problem: there are some operators which are not in general positive or linear such as, Picard, Gauss-Weierstrass, Poisson-Cauchy Operators.
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Density of a Set

Let $K \subset \mathbb{N}$. Define

$$K_n = \{k \leq n : k \in K\},$$

then we give the density of the set $K$ as

$$\delta(K) := \lim_{n} \frac{\# \{K_n\}}{n}.$$

For example,

$$\delta(\mathbb{N}) = 1.$$
Let \( (x_k)_{k \in \mathbb{N}} \) be a sequence of real or complex numbers, then

\[
st - \lim_{k} x_k = L \iff \forall \epsilon > 0, \delta(\{k : |x_k - L| \geq \epsilon\}) = 0.
\]
Let \((x_k)_{k \in \mathbb{N}}\) be a sequence of real or complex numbers, Then

\[
st - \lim_{k} x_k = L \iff \forall \epsilon > 0, \; \delta(\{k : |x_k - L| \geq \epsilon\}) = 0.
\]

Equivalently, let \(K(\epsilon) := \{k \leq n : |x_k - L| > \epsilon\}\). Then

\[
st - \lim_{k} x_k = L \iff \forall \epsilon > 0,
\]

\[
\lim_n \left( C_1 \chi_{K(\epsilon)} \right)_n := \lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_{K(\epsilon)}(k) = 0
\]

where \(C_1 := (c_{n,k})\) is the Cesáro matrix.
An Example

Let

\[ x_n = \begin{cases} \sqrt{n}, & \text{if } n = k^2, \; k = 1, 2, \ldots \\ 0, & \text{otherwise}. \end{cases} \]

\((x_n)_{n \in \mathbb{N}}\) is not convergent.
An Example

Let

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\( (x_n)_{n \in \mathbb{N}} \) is not convergent.

\( (x_n)_{n \in \mathbb{N}} \) does not have an upper bound.
Let

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- \((x_n)_{n\in\mathbb{N}}\) is not convergent.
- \((x_n)_{n\in\mathbb{N}}\) does not have an upper bound.
- \(\text{st } \lim_n x_n = 0.\)
Let $A = (a_{nk})$ be non-negative regular matrix. Let 
$K(\epsilon) := \{k \leq n : |x_k - L| > \epsilon\}$. Then,

$$st_A \lim_{k} x_k = L \iff \forall \epsilon > 0,$$

$$\lim_{n} \sum_{k=1}^{n} a_{nk} \chi_{K(\epsilon)}(k) = 0,$$

or equivalently,

$$\lim_{n} \sum_{k : |x_k - L| \geq \epsilon} a_{nk} = 0.$$

- $A = I \rightarrow$ classical convergence,
- $A = C_1 \rightarrow$ statistical convergence.
Definition of The Operators

For \( r \in \mathbb{N} \) and \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), we define

\[
\alpha^{[m]}_{j,r} = \begin{cases} 
(-1)^{r-j} \binom{r}{j} j^{-m}, & j = 1, \ldots, r, \\
1 - \sum_{k=1}^{r} (-1)^{r-k} \binom{r}{k} k^{-m}, & j = 0
\end{cases}
\]

and

\[
\delta^{[m]}_{k,r} = \sum_{j=1}^{r} \alpha^{[m]}_{j,r} j^k; \text{ where } k = 1, 2, \ldots, m \in \mathbb{N}.
\]

Observe that

\[
\sum_{j=0}^{r} \alpha^{[m]}_{j,r} = 1.
\]
Definition of The Operators

Now let $f \in C^m(\mathbb{R})$, $x \in \mathbb{R}$, $n, r \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $(\zeta_n)_{n \in \mathbb{N}}$ be any sequence of real numbers such that $0 < \zeta_n \leq 1$.

- We define the generalized unitary discrete Picard operators as:

$$
P^{[m]}_{r, \zeta_n}(f; x) := \frac{\sum_{\nu=-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{-|\nu| \zeta_n}}{\sum_{\nu=-\infty}^{\infty} e^{-|\nu| \zeta_n}}
$$
Definition of The Operators

Now let \( f \in C^m(\mathbb{R}) \), \( x \in \mathbb{R} \), \( n, r \in \mathbb{N} \), \( m \in \mathbb{N}_0 \), and \( (\xi_n)_{n \in \mathbb{N}} \) be any sequence of real numbers such that \( 0 < \xi_n \leq 1 \),

- We define the generalized unitary discrete Picard operators as:

\[
P_{r,\xi_n}^*[m](f; x) :\begin{align*}
&= \sum_{\nu=-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{-|\nu|} \\
&= \sum_{\nu=-\infty}^{\infty} e^{-|\nu|} \xi_n
\end{align*}
\]

- We define the generalized unitary discrete Gauss-Weierstrass operators as:

\[
W_{r,\xi_n}^*[m](f; x) :\begin{align*}
&= \sum_{\nu=-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{-\nu^2} \\
&= \sum_{\nu=-\infty}^{\infty} e^{-\nu^2} \xi_n
\end{align*}
\]
Definition of The Operators

\( f \in C^m(\mathbb{R}) \), \( x \in \mathbb{R} \), \( n, r \in \mathbb{N} \), \( m \in \mathbb{N}_0 \), and \( (\xi_n)_{n \in \mathbb{N}} \) be any sequence of real numbers such that \( 0 < \xi_n \leq 1 \),

- Let \( s \in \mathbb{N} \) and \( t > \frac{1}{s} \). We define the generalized unitary discrete Poisson-Cauchy operators as:

\[
\theta^*[m]_{r,\xi_n}(f; x) := \frac{\sum_{\nu=-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j \nu) \right) (\nu^{2s} + \xi^{2s})^{-t}}{\sum_{\nu=-\infty}^{\infty} (\nu^{2s} + \xi^{2s})^{-t}}
\]
Definition of The Operators

\( f \in C^m(\mathbb{R}), \ x \in \mathbb{R}, \ n, r \in \mathbb{N}, \ m \in \mathbb{N}_0, \) and \((\xi_n)_{n \in \mathbb{N}}\) be any sequence of real numbers such that \(0 < \xi_n \leq 1,\)

- Let \(s \in \mathbb{N}\) and \(t > \frac{1}{s}.\) We define the generalized unitary discrete Poisson-Cauchy operators as:

\[
\theta^*[m]_{r, \xi_n}(f; x) := \sum_{\nu = -\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) \left( \nu^{2s} + \xi^{2s} \right)^{-t}
\]

- Observe that for \(c\) constant we have

\[
P^*[m]_{r, \xi_n}(c; x) = W^*[m]_{r, \xi_n}(c; x) = \theta^*[m]_{r, \xi_n}(c; x) = c.
\]
Definition of The Operators

\( f \in C^m(\mathbb{R}) \), \( x \in \mathbb{R}, n, r \in \mathbb{N}, m \in \mathbb{N}_0, \) and \( (\tilde{\xi}_n)_{n \in \mathbb{N}} \) be any sequence of real numbers such that \( 0 < \tilde{\xi}_n \leq 1 \),

- Let \( s \in \mathbb{N} \) and \( t > \frac{1}{s} \). We define the generalized unitary discrete Poisson-Cauchy operators as:

\[
\theta_{r,\tilde{\xi}_n}^*[m](f; x) := \frac{\sum_{\nu=-\infty}^{\infty} \left( \sum_{j=0}^{r} \alpha_j f(x + j\nu) \right) (\nu^{2s} + \tilde{\xi}^{2s})^{-t}}{\sum_{\nu=-\infty}^{\infty} (\nu^{2s} + \tilde{\xi}^{2s})^{-t}}
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- Observe that for \( c \) constant we have

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\]

- We assume that the operators \( P_{r,\tilde{\xi}_n}^*[m](f; x), W_{r,\tilde{\xi}_n}^*[m](f; x) \), and \( \theta_{r,\tilde{\xi}_n}^*[m](f; x) \in \mathbb{R} \), for \( x \in \mathbb{R} \). This is the case when \( \|f\|_{\infty,\mathbb{R}} < \infty \).
The operators $P^*[m]_{r,\xi_n}$, $W^*[m]_{r,\xi_n}$, and $\theta^*[m]_{r,\xi_n}$ are not necessarily positive linear operators.
The operators $P_{r,\xi}^{*[m]}$, $W_{r,\xi}^{*[m]}$, and $\theta_{r,\xi}^{*[m]}$ are not necessarily positive linear operators.

Let $r = 2$, $m = 3$, $x = 0$, and $f(t) = t^2 \geq 0$, we have $\alpha_{0,2}^{[3]} = \frac{23}{8}$, $\alpha_{1,2}^{[3]} = -2$, and $\alpha_{2,2}^{[3]} = \frac{1}{8}$. Then

$$P_{2,\xi}^{*[3]}(t^2;0) < 0$$

$$W_{2,\xi}^{*[3]}(t^2;0) < 0$$

$$\theta_{2,\xi}^{*[3]}(t^2;0) < 0$$
Some Necessary Material

- \textit{rth} modulus of smoothness finite given as

\[
\omega_r(f^{(m)}, h) := \sup_{|t| \leq h} \| \Delta_t^r f^{(m)}(x) \|_{\infty, x} < \infty, \ h > 0, 
\]

where \( \| . \|_{\infty, x} \) is the supremum norm with respect to \( x \), \( f \in C^m(\mathbb{R}) \), \( m \in \mathbb{N}_0 \), and

\[
\Delta_t^r f^{(m)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(m)}(x + jt).
\]
Some More Necessary Material

Proposition (Anastassiou-Kester, 2015)

Let \( m, r \in \mathbb{N} \). Then

\[
0 \leq \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi_n}\right)^r e^{-|\nu|/\xi_n} < \sum_{\nu=-\infty}^{\infty} \frac{|\nu|^m \left(1 + |\nu|/\xi_n\right)^r e^{-|\nu|/\xi_n}}{e^{j\nu/\xi_n}} < K_{r,m}
\]

and

\[
0 \leq c_{k, \xi_n}^* := \sum_{\nu=-\infty}^{\infty} \nu^k e^{-|\nu|/\xi_n} < K_{r,m}, \quad k = 1, \ldots, m
\]

for all \((\xi_n)_{n \in \mathbb{N}}\) such that \(0 < \xi_n \leq 1, \forall n \in \mathbb{N}\). Here, \(K_{r,m}\) is a constant depending on \(r\) and \(m\).
Theorem (Anastassiou-Kester, 2015)

Let \( f \in C^m(\mathbb{R}) \) with \( f^{(m)} \in C_u(\mathbb{R}) \) (uniformly continuous functions), \( m \in \mathbb{N} \). Then

\[
\left\| P_{r,\xi_n}^*[m] (f; x) - f(x) - \sum_{k=1}^{m} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi_n}^* \right\|_{\infty, x} \leq K_{r,m} \frac{\omega_r(f^{(m)}, \xi_n)}{m!},
\]

\( K_{r,m} \) is a constant depending on \( r \) and \( m \).
Uniform Approximation Results

Theorem (Anastassiou-Kester, 2015)

Let \( f \in C_u(\mathbb{R}) \). Then

\[
\left\| P_{r,\xi_n}^* (f; x) - f(x) \right\|_{\infty, x} \leq K_{r,m} \omega_r(f, \xi_n),
\]

\( K_{r,m} \) is a constant depending on \( r \) and \( m \).
(\(m \in \mathbb{N}\)) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r, m \in \mathbb{N}\) and for every \(f \in C^m(\mathbb{R})\) with \(f^{(m)} \in C_u(\mathbb{R})\)

\[
\left\| P_{r, \xi_n}^{[m]} (f; x) - f(x) - \sum_{k=1}^{m} \frac{f^{(k)}(x)}{k!} \delta_k c_{k, \xi_n}^{*} \right\|_{\infty, x} \to 0 \text{ as } \xi_n \to 0^+
\]
What does it really mean?

1. \((m \in \mathbb{N})\) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r, m \in \mathbb{N}\) and for every \(f \in C^m(\mathbb{R})\) with \(f^{(m)} \in C_u(\mathbb{R})\)

\[
\left\| P_{r, \xi_n}^{[m]} (f; x) - f(x) - \sum_{k=1}^{m} \frac{f^{(k)}(x)}{k!} \delta_k c_{k, \xi_n} \right\|_{\infty, x} \rightarrow 0 \quad \text{as} \quad \xi_n \rightarrow 0^+
\]

2. \((m = 0)\) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r \in \mathbb{N}\) and for every \(f \in C_u(\mathbb{R})\)

\[
\left\| P_{r, \xi_n}^{[0]} (f; x) - f(x) \right\|_{\infty, x} \rightarrow 0 \quad \text{as} \quad \xi_n \rightarrow 0^+
\]
(1) (\(m \in \mathbb{N}\)) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r, m \in \mathbb{N}\) and for every \(f \in C^m(\mathbb{R})\) with \(f^{(m)} \in C_u(\mathbb{R})\)

\[
\left\| P_{r, \xi_n}^{*[m]} (f; x) - f(x) - \sum_{k=1}^{m} \frac{f^{(k)}(x)}{k!} \delta_k c_{k, \xi_n}^* \right\|_{\infty, x} \to 0 \text{ as } \xi_n \to 0^+
\]

(2) \((m = 0)\) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r \in \mathbb{N}\) and for every \(f \in C_u(\mathbb{R})\)

\[
\left\| P_{r, \xi_n}^{*[0]} (f; x) - f(x) \right\|_{\infty, x} \to 0 \text{ as } \xi_n \to 0^+
\]

(3) Can we get more powerful results than these?
Theorem (Kester, 2018)

Let \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), \(f \in C^m(\mathbb{R})\) with \(f^{(m)} \in C_u(\mathbb{R})\) such that

\[
\left\| f^{(m)}(x) \right\|_{\infty, x} < \infty
\]

for every \(m \in \mathbb{N}\). Then

\[
\left\| P_{r, \xi_n}^{[m]}(f; x) - f(x) \right\|_{\infty, x} \leq C_{r,m} \omega_r(f^{(m)}, \xi_n) + T_{r,m} e^{-\frac{1}{\xi_n}}
\]

where \(C_{r,m}\) and \(T_{r,m}\) are constants depending on \(r\) and \(m\).
(\(m \in \mathbb{N}\)) When \((\xi_n)_{n \in \mathbb{N}}\) is a sequence of real numbers such that \(0 < \xi_n \leq 1\) for every \(n \in \mathbb{N}\), then for each fixed \(r, m \in \mathbb{N}\) and for every \(f \in C^m(\mathbb{R})\) with \(f^{(m)} \in C_u(\mathbb{R})\) and

\[
\left\| f^{(m)}(x) \right\|_{\infty, x} < \infty,
\]

we have

\[
\left\| P_{r, \xi_n}^{[m]}(f; x) - f(x) \right\|_{\infty, x} \to 0 \text{ as } \xi_n \to 0^+.
\]
Statistical Approximation Results

**Theorem (Kester, 2018)**

Let \( m, r \in \mathbb{N} \) and \( A = [a_{j,n}] \) be a non-negative regular summability matrix, and let \( (\xi_n)_{n \in \mathbb{N}}, 0 < \xi_n \leq 1 \), be a sequence of positive real numbers for which

\[
\text{st}_A \lim_{n} \xi_n = 0.
\]

Then, for each fixed \( m, r \in \mathbb{N} \) and for every \( f \in C^m(\mathbb{R}) \) with \( f^{(m)} \in C_u(\mathbb{R}) \) such that

\[
\left\| f^{(m)}(x) \right\|_{\infty,x} < \infty,
\]

we have

\[
\text{st}_A \lim_{n} \left\| P^{[m]}_{r,\xi_n}(f;x) - f(x) \right\|_{\infty,x} = 0.
\]
Theorem (Kester, 2018)

Let \( m, r \in \mathbb{N} \) and \( A = [a_{j,n}] \) be a non-negative regular summability matrix, and let \( (\xi_n)_{n \in \mathbb{N}} \), \( 0 < \xi_n \leq 1 \), be a sequence of positive real numbers for which

\[
\text{st}_A \lim_{n} \xi_n = 0.
\]

Then, for each fixed \( r \in \mathbb{N} \) and for every \( f \in C_u(\mathbb{R}) \), we have

\[
\text{st}_A \lim_{n} \left\| P^{[0]}_{r,\xi_n}(f; x) - f(x) \right\|_{\infty, x} = 0.
\]
More Powerful? Why?

- $A = c_1,$
More Powerful? Why?

- $A = C_1$,
- Let $(\xi_n)_{n \in \mathbb{N}}$ be defined as

$$\xi_n = \begin{cases} \frac{n}{n+1}, & \text{if } n = k^2, \ k = 1, 2, \ldots \\ 1/n, & \text{otherwise}. \end{cases}$$
\begin{itemize}
\item $A = C_1$, 
\item Let $(\xi_n)_{n \in \mathbb{N}}$ be defined as 
\[ \xi_n = \begin{cases} 
\frac{n}{n+1}, & \text{if } n = k^2, \ k = 1, 2, \ldots \\
1/n, & \text{otherwise.} 
\end{cases} \]
\item 
\[ P_{r,\xi_n}^*[m] (f; x) \rightarrow f, \ \text{as } \xi_n \rightarrow 0^+ \]
\end{itemize}
However,

\[ st_{C_1} - \lim_{n} \xi_n = 0, \]

which gives us

\[ st_{C_1} - \lim_{n} \left\| P^{[m]}_{r, \xi_n} (f; x) - f(x) \right\|_{\infty, x} = 0. \]

for each fixed \( m \in \mathbb{N}_0. \)
What can be done next?

- Can we get statistical approximation results for non-unitary versions of $P_{r,\xi_n}^*[m]$, $W_{r,\xi_n}^*[m]$, and $\theta_{r,\xi_n}^*[m]$?
- $L_p$ norm?
- New operators, new approximations.