Compression & compact perturbation of operators
A numerical range approach

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Let $B(H)$ be the algebra of bounded linear operators acting on the Hilbert space $H$ equipped with the inner product $\langle x, y \rangle$. 

Basic problems

1. Find a unitary $U \in B(H)$ such that $U^*A_jU = (T_j^* \cdots T_1^*)$, $j = 1, \ldots, m$, for some desirable $T_1, \ldots, T_m$.

2. Find those "good" compression $T_1, \ldots, T_m$ such that for any low rank, finite rank, or compact operators $K_1, \ldots, K_m$, there is an isometry $X$ (depending on $K_j$'s) satisfying $T_j = X^*(A_j + K_j)X$, $j = 1, \ldots, m$. 

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The joint numerical range of $A = (A_1, \ldots, A_m) \in B(H)^m$ is defined by
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Let $A = A_1 + iA_2 \in B(H)$, where $A_1 = A_1^*$ and $A_2 = A_2^*$. Then

$$W(A) \equiv W(A_1, A_2) = \{ (\langle A_1 x, x \rangle, \langle A_2 x, x \rangle) : x \in H, \langle x, x \rangle = 1 \} \subseteq \mathbb{R}^2.$$
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The joint numerical ranges $W(A_1, \ldots, A_k)$ are all possible measurements of pure states using the measurement operators (observable) $A_1, \ldots, A_k$. 

(Au-Yeung and Poon, 1979) If $n \geq 3$ and $A = (A_1, A_2, A_3) \in M_n$ is a triple of Hermitian matrices, then $W(A)$ is convex. If $m \geq 4$, then $W(A_1, \ldots, A_m)$ may not be convex even if $\dim H = \infty$. 

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If $m \geq 4$, then $W(A_1, \ldots, A_m)$ may not be convex even if $\dim H = \infty$. 
In quantum information science, quantum channels/operations are trace preserving completely positive linear maps $\Phi : M_n \rightarrow M_m$ admitting the operator sum representation

$$\Phi(X) = F_1XF_1^* + \cdots + F_rXF_r^*$$

for some $F_1, \ldots, F_r$ satisfying $\sum_{j=1}^{r} F_j^*F_j = I_n$. 
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There will be a recovery channel $\Psi$ such that

$$\Psi \circ \Phi(X) = X \quad \text{whenever} \quad PXP = X,$$

where $P$ is the orthogonal projection of $H$ onto the “coding” subspace.
The joint higher rank numerical range was introduced in [Choi, Kribs, Zyczkowski, 2006].

Note that \((a_1, \ldots, a_m) \in \Lambda^p(A)\) if and only if there is a unitary 
\[ U = \begin{bmatrix} X & \tilde{X} \end{bmatrix} \]
such that 
\[ U^* A U = (a_j I_p)^{\ast \ast \ast}, \quad j = 1, \ldots, m. \]

Suppose \(A_1\) has eigenvalues \(\lambda_1, \ldots, \lambda_n\). Then 
\[ \Lambda^p(A_1) = [\lambda_{n-p+1}, \lambda_p]. \]

The set \(\Lambda^p(A_1)\) may be empty if \(p > (\dim H + 1)/2\).

If \(\Lambda^p(A)\) is convex, one can derive efficient algorithms to find its elements, and construct quantum error correction codes accordingly.

However, one only has convexity if \(m \leq 2\). [Choi et al., 2006], [Woerdeman, 2009], [Li and Sze, 2009]

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**Theorem [Li and Tsing, 1991]**

Let \( A \in M_n \) be Hermitian matrix with eigenvalues \( a_1 \geq \cdots \geq a_n \). Then the set \( W(q : A) \) consists of Hermitian matrices \( B \in M_q \) with eigenvalues

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Consequently, the set $W(q : A)$ is convex if and only if

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- In general, the structure of $W(q : A)$ is hard to determine.
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$$U^* A_j U = \begin{pmatrix} B_j & * \\ * & * \end{pmatrix}, \quad j = 1, \ldots, m.$$

**Theorem [Li and Tsing, 1991]**

Let $A \in M_n$ be Hermitian matrix with eigenvalues $a_1 \geq \cdots \geq a_n$. Then the set $W(q : A)$ consists of Hermitian matrices $B \in M_q$ with eigenvalues $b_1 \geq \cdots \geq b_q$ satisfying $a_j \geq b_j \geq a_{n-q+j}$ for $j = 1, \ldots, q$.

Consequently, the set $W(q : A)$ is convex if and only if

$$a_1 = \cdots = a_q \quad \text{and} \quad a_{n-q+1} = \cdots = a_n.$$ 

- In general, the structure of $W(q : A)$ is hard to determine.
- For example, even for a normal matrix $A \in M_n$, it is not easy to characterize those (normal) matrices in $W(q : A)$. 

To construct a “better” quantum error correction, researchers consider the $(p, q)$-matricial range of $A = (A_1, \ldots, A_m)$ to be the set $\Lambda_{p,q}(A)$ of $m$-tuple of Hermitian matrices $(B_1, \ldots, B_m)$ such that

$$X^* A_j X = I_p \otimes B_j = B_j \oplus \cdots \oplus B_j \ (p \text{ times}), \quad j = 1, \ldots, m,$$

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\ast & \ast & \ast \\
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Evidently, this definition covers the joint numerical range, the joint rank \(p\)-numerical range, the \(q\)-matricial range as special cases.
While there is no general convexity result for $\Lambda_{p,q}(A)$, we may obtain some star-shapedness results.
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Recall that a set $S \subseteq \mathbb{R}^N$ is **star-shaped** if there is a star center $v_0 \in S$ such that the line segment joining $v_0$ to any other point $v \in S$ lie in $S$. 

**Theorem** [Lau, Li, Poon, Sze, 2018]

Let $A = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$, $p$ and $q$ be positive integers.

(a) If $\dim \mathcal{H} \geq (pq - 1)(m + 1)^2$, then $\Lambda_{pq}(A)$ and $\Lambda_{p,q}(A)$ are non-empty.

(b) If $\dim \mathcal{H} \geq (N - 1)(m + 1)^2$ for $N = pq(m + 2)$, then every element in $\text{conv}\{ (a_{1Iq}, \ldots, a_{mIq}) : (a_1, \ldots, a_m) \in \Lambda_N(A) \} \subseteq \Lambda_{p,q}(A)$ is a star center of $\Lambda_{p,q}(A)$.

(c) For any $r$ with $1 \leq qr < p \leq \dim \mathcal{H}$, if $K = (K_1, \ldots, K_m)$ with $K_1, \ldots, K_m \in \mathcal{B}(\mathcal{H})$ such that $\text{rank}(K_2^1 + \cdots + K_{2m}) \leq r$, then $\Lambda_{p,q}(A) \subseteq \Lambda_{p-qr,q}(A + K)$. 

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Nonemptyness, Star-shapedness, low rank perturbations

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Let $\mathbf{A} = (A_1, \ldots, A_m)$ be an $m$-tuple of self-adjoint operators in $B(H)^m$. The set

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\[
(a_1 I_1, \ldots, a_m I_q) \quad \text{is a star center whenever} \quad (a_1, \ldots, a_m) \in \text{conv} \ \Lambda_N(\mathbf{A}).
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More results when \( \dim H \) is infinite

Suppose \( \dim H = \infty \). We consider

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\Lambda_\infty(A) = \bigcap_{r \in \mathbb{N}} \Lambda_r(A) \subseteq \mathbb{R}^m \quad \text{and} \quad \Lambda_\infty,q(A) = \bigcap_{r \in \mathbb{N}} \lambda_{r,q}(A) \subseteq M^m_q.
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**Example** If $A = \text{diag}(1, 1/2, 1/3, \ldots)$, then for any positive integer $q$, $\Lambda_\infty,q(A) = \emptyset$. 
Let $K(H)$ is the set of compact operators in $B(H)$. Define the essential $(p, q)$-matricial range by

$$\Lambda_{p,q}^{\text{ess}}(A) = \cap \{ \text{cl} (\Lambda_{p,q}(A+K)) : K \in K(H)^m \}.$$ 

When $p = 1$, we get the essential $q$-matricial range

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$$\Lambda^{\text{ess}}_{p,q}(A) = W_{\text{ess}}(q : A)$$

is compact, convex, and non-empty.
Further research

- Find the smallest $\dim H$ that ensures $\Lambda_p(A) \neq \emptyset$ for all $A \in B(H)^m$. 

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Compression and compact perturbation of operators
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  \[ \text{cl}(W(q : A + K)) = W_{\text{ess}}(q : A). \]
News on Workshop on Numerical Ranges and Numerical Radii

Welcome to the wonderland of numerical range!
The 14th workshop on Numerical Range and Numerical Radii will be held at Munich, June 13-17, 2018.
To celebrate the 100 year anniversary of the Töplitz-Hausdorff Theorem, there will be a 2019 workshop at Kyoto/Osaka area, Japan.
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