INVITATION TO LINEAR PRESERVER PROBLEMS, PART II

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I. Generalized inverse preservers maps

Definition An element $b \in \mathcal{A}$ is called a generalized inverse of $a \in \mathcal{A}$ if $b$ satisfies the following two identities

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- If \( \mathcal{A} = \mathcal{A}^\wedge \) then \( \mathcal{A} \) is finite dimensional.
- Furthermore, if \( \mathcal{A} \) is semi-simple, then
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In particular, for the special case of the complex matrix algebra $A = M_n(\mathbb{C})$, we have $A = A^\wedge$. 
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Let \( H \) be an infinite-dimensional separable complex Hilbert space and \( B(H) \) the algebra of all bounded linear operators on \( H \) and \( K(H) \subset B(H) \) be the closed ideal of all compact operators. We denote the Calkin algebra \( B(H)/K(H) \) by \( C(H) \). Let \( \pi : B(H) \to C(H) \) be the quotient map.
Generalized inverse

Theorem [M. Mbekhta, L. Rodman and P. Šemrl, 2006]

Let $H$ be an infinite-dimensional separable Hilbert space and let $\phi : B(H) \rightarrow B(H)$ be a bijective continuous unital linear map preserving generalized invertibility in both directions. Then

$$\phi(K(H)) = K(H),$$

and the induced map $\varphi : C(H) \rightarrow C(H)$, (i.e. $\varphi \circ \pi = \pi \circ \phi$), is either an automorphism, or an anti-automorphism.
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Can we relax the assumptions of this theorem? The following theorem, answers this question in the affirmative.
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We denote by $SF(H) \subset B(H)$ the subset of all semi-Fredholm operators and let $F(H) \subset B(H)$ the ideal of all finite rank operators,
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**Definition** We say that $\phi : B(H) \rightarrow B(H)$ is surjective up to finite rank (resp. compact) operators if for every $A \in B(H)$ there exists $B \in B(H)$ such that $A - \phi(B) \in F(H)$ (resp. $K(H)$).
Theorem [M. Mbekhta and P. Šemrl 2009]

Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a surjective up to finite rank operators linear map.

If $\phi$ preserves generalized invertibility in both directions, then $\phi(K(H)) \subseteq K(H)$ and the induced map $\phi : C(H) \to C(H)$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in C(H)$.

The above theorem allows us to establish the following result which is of independent interest.

We say that a map $\phi : B(H) \to B(H)$ preserves semi-Fredholm operators in both directions if for every $A \in B(H)$ the operator $\phi(A)$ is semi-Fredholm if and only if $A$ is.
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*Under the same hypothesis and notation as in the above theorem, the following statements hold:*

1. $\varphi$ preserves Fredholm operators in both directions;
2. There is an $n_0 \in \mathbb{Z}$ such that either $\text{ind}(\varphi(T)) = n_0 + \text{ind}(T)$ for every Fredholm operator $T$, or $\text{ind}(\varphi(T)) = n_0 - \text{ind}(T)$ for every Fredholm operator $T$. 
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Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a surjective up to compact operators linear map. If $\phi$ preserves semi-Fredholm operators in both directions, then $\phi(K(H)) \subseteq K(H)$ and the induced map $\varphi : C(H) \to C(H)$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in C(H)$.

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**INVITATION TO LINEAR PRESERVER PROBLEMS, PART II**
Lemma

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Corollary

Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : B(H) \rightarrow B(H)$ a surjective up to finite rank op linear map. Then $\phi$ preserves generalized invertibility in both directions implies $\phi$ preserves semi-Fredholm operators in both directions.
For $T \in \mathcal{B}(H)$, the \textit{essential spectrum}, $\sigma_e(T)$, of $T$, is defined as the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{C}(H)$, i.e. $\sigma_e(T) = \sigma(\pi(T))$. Obviously,

$$\sigma_e(T) = \{ \lambda \in \mathbb{C}; \quad T - \lambda I \text{ is not Fredholm} \}.$$
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We say that a linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves the essential spectrum if $\sigma_e(\phi(T)) = \sigma_e(T)$ for all $T \in \mathcal{B}(H)$. 

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**Theorem**

Let $H$ be an infinite-dimensional Hilbert space and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators. Then the following are equivalent:

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(ii) $\phi$ preserves the set of Fredholm operators in both directions and $\phi(I) = I - K$ where $K \in \mathcal{K}(H)$;

(iii) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and the induced map $\phi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\phi \circ \pi = \pi \circ \phi$, is either an automorphism, or an anti-automorphism.
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(iii) $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi \circ \pi = \pi \circ \phi$, is either an automorphism, or an anti-automorphism.
Conjecture

Let $H$ be an infinite-dimensional Hilbert space and let $\phi : B(H) \to B(H)$ be a linear map. Assume that $\phi$ is surjective up to compact operators. Then the following conditions are equivalent:

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(III) either (i) $\phi(T) = ATA^{-1} + \chi(T)$ for every $T \in B(H)$ where $A$ is an invertible operator in $B(H)$ and $\chi : B(H) \to K(H)$ linear, or (ii) $\phi(T) = BT_{tr}B^{-1} + \chi(T)$ for every $T \in B(H)$ where $B$ is an invertible operator in $B(H)$ and $\chi : B(H) \to K(H)$ linear.

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We denote by $A^{-1}$ the set of invertible elements of $A$. We shall say that an additive map $\phi: A \to B$ strongly preserves invertibility if $\phi(x^{-1}) = \phi(x)^{-1}$ for every $x \in A^{-1}$. Similarly, we shall say that $\phi$ strongly preserves generalized invertibility if $\phi(y)$ is a generalized inverse of $\phi(x)$ whenever $y$ is a generalized inverse of $x$.

Remark. One easily checks that a Jordan homomorphism strongly preserves invertibility (resp. generalized inverses). The motivation for this problem is Hua's theorem which states that every unital additive map $\phi$ between two fields such that $\phi(x^{-1}) = \phi(x)^{-1}$ is an isomorphism or an anti-isomorphism.
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**Theorem [N.Boudi and M.M. 2010]**

Let $A$ and $B$ be unital Banach algebras and let $\phi : A \to B$ be an additive map.

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**Corollary**

Let $\phi : M_n(C) \to M_n(C)$, be a linear map. Then the following conditions are equivalent:

1. $\phi$ preserves invertibility;
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INVITATION TO LINEAR PRESERVER PROBLEMS, PART II*
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INVITATION TO LINEAR PRESERVER PROBLEMS, PART II
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IV. Moore-Penrose inverses preservers maps

In the context $C^*$-algebras, it is well known that every generalized invertible element $a$ has a unique generalized inverse $b$ for which $ab$ and $ba$ are projections, such an element $b$ is called the Moore-Penrose inverse of $a$ and denoted by $a^\dagger$. 
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In other words, $a^\dagger$ is the unique element of $A$ that satisfies:

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We will say that a linear map $\phi : A \to B$ is $C^*$-Jordan homomorphism if it is a Jordan homomorphism which preserves the adjoint operation, i.e. $\phi(x^*) = \phi(x)^*$ for all $x$ in $A$. 
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The $C^*$-homomorphism and $C^*$-anti-homomorphism are analogously defined.
Theorem [M.M.]

Let $A$ be a $C^*$-algebra of real rank zero and $B$ a prime $C^*$-algebra. Let $\phi : A \to B$ be a surjective, unital linear map. Then the following conditions are equivalent:

1) $\phi(x^\dagger) = \phi(x)^\dagger$ for all $x \in A^\dagger$;

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Denote by $B^\dagger(H)$ the set of the operators on $H$ that possess a Moore-Penrose inverse.

Corollary

Let $\phi : B(H) \to B(H)$ be a surjective unital additive map. Then the following conditions are equivalent:

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INVITATION TO LINEAR PRESERVER PROBLEMS, PART II
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Denote by $B^\dagger(H)$ the set of the operators on $H$ that possess a Moore-Penrose inverse.

Corollary

Let $\phi : B(H) \to B(H)$ be a surjective unital additive map. Then the following conditions are equivalent:

(1) $\phi(T^\dagger) = \phi(T)^\dagger$ for all $T \in B^\dagger(H)$;
(2) there is a unitary operator $U$ in $B(H)$ such that $\phi$ takes one of the following forms
\[ \phi(T) = UTU^* \quad \text{or} \quad \phi(T) = UT^{tr}U^* \quad \text{for all } T. \]
In connection with Theorem, we conclude by the following conjecture

**Conjecture**

Let $A$ and $B$ be $C^*$-algebras. Let $\phi: A \rightarrow B$ be a surjective linear map. Then the following conditions are equivalent:

1) $\phi(x^\dagger) = \phi(x)^\dagger$ for all $x \in A^\dagger$;
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V. ascent and descent preserver maps
The ascent $a(T)$ and descent $d(T)$ of $T \in \mathcal{L}(X)$ are defined by

\[ a(T) = \inf\{n \geq 0 : \ker(T^n) = \ker(T^{n+1})\} \]
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where the infimum over the empty set is taken to be infinite.

An operator $T \in \mathcal{L}(X)$ is said to have a Drazin inverse, or to be Drazin invertible, if there exists $S \in \mathcal{L}(X)$ and a non-negative integer $n$ such that

$$T^{n+1}S = T^n, \quad STS = S \quad \text{and} \quad TS = ST.$$  \hspace{1cm} (1)

Note that if $T$ possesses a Drazin inverse, then it is unique and the smallest non-negative integer $n$ in (1) is called the index of $T$ and is denoted by $i(T)$. It is well known that $T$ is Drazin invertible if and only if it has finite ascent and descent, and in this case $a(T) = d(T) = i(T)$. 
Recall also that an operator \( T \in \mathcal{L}(X) \) is called \textit{upper} (resp. \textit{lower}) \textit{semi-Fredholm} if \( \text{R}(T) \) is closed and \( \dim \text{N}(T) \) (resp. \( \text{codim} \text{R}(T) \)) is finite. The set of such operators is denoted by \( \mathcal{F}_+(X) \) (resp. \( \mathcal{F}_-(X) \)). The \textit{Fredholm} and \textit{semi-Fredholm} subsets are defined by \\
\( \mathcal{F}(X) := \mathcal{F}_+(X) \cap \mathcal{F}_-(X) \) and \( \mathcal{F}_\pm(X) := \mathcal{F}_+(X) \cup \mathcal{F}_-(X) \), respectively.
Let us introduce the following subsets:

(i) $A(X) := \{ T \in \mathcal{L}(X) : a(T) < \infty \}$ the set of finite ascent operators,
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(iii) \( \mathcal{D}^r(X) := \mathcal{A}(X) \cap \mathcal{D}(X) \) the set of Drazin invertible operators,

(iv) \( \mathcal{B}^+ (X) := \mathcal{F}^+ (X) \cap \mathcal{A}(X) \) the set of upper semi-Browder operators,

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Let \( S \) denotes one of the subsets (i)-(vii). A surjective additive maps \( \Phi : \mathcal{L}(X) \to \mathcal{L}(Y) \) is said to preserve \( S \) in the both direction if \( T \in S \) if and only if \( \Phi(T) \in S \).
Theorem [M.M, V.Muller, M.Oudghiri 2014]

Let $\Phi : \mathcal{L}(H) \rightarrow \mathcal{L}(K)$ be a surjective additive continuous map. Then the following assertions are equivalent:

(i) $\Phi$ preserves the finiteness-of-ascent.

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(iii) $\Phi$ preserves in both direction $B^+$.

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(v) There exists an invertible bounded linear, or conjugate linear, operator $A : H \rightarrow K$ and a non-zero complex number $c$ such that $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(H)$.
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INVITATION TO LINEAR PRESERVER PROBLEMS, PART II
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Mostafa Mbekhta
INVITATION TO LINEAR PRESERVER PROBLEMS, PART II
Definition

$T \in \mathcal{L}(X)$ is said to be \textit{group invertible} if there exists $S \in \mathcal{L}(X)$ such that

$$TST = T, STS = S \text{ and } ST = TS.$$ 

In this case, $S$ is \textit{unique} and is denoted by $T^\#$. 

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\( T \in \mathcal{L}(X) \) is said to be *group invertible* if there exists \( S \in \mathcal{L}(X) \) such that

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**INVITATION TO LINEAR PRESERVER PROBLEMS, PART II**
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- \( \text{Inv}(ℒ(𝑋)) \subseteq G \), in this case *T# = T}^{-1} \) for *T* invertible.
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- $T \in \mathcal{G} \iff T^* \in \mathcal{G}(X^*)$. Furthermore, $(T^*)^\# = (T^\#)^*$. 
**Definition**

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- $T \in \mathcal{G} \leftrightarrow T$ is Drazin invertible with $i(T) \leq 1$.
- $T \in \mathcal{G} \leftrightarrow T^* \in \mathcal{G}(X^*)$. Furthermore, $(T^*)^\# = (T^\#)^*$.
- $T \in \mathcal{G} \leftrightarrow \exists P; P^2 = P$ and $T + P$ is invertible.
Theorem [M.M-Oudghiri, 2017]

Let $X$ be a real or complex infinite-dimensional Banach space, and let $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. The following assertions are equivalent.

(i) $\Phi$ preserves $G$ in both directions;

(ii) there exist a non-zero $\alpha \in \mathbb{C}$ and a bounded invertible linear, or conjugate linear, operator $A$ between suitable spaces such that $\Phi(T) = \alpha ATA^{-1}$ for all $T \in \mathcal{L}(X)$ or $\Phi(T) = \alpha AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$. 
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Question

It would be interesting to know if the main theorem holds true for surjective additive maps $\Phi$ on $\mathcal{L}(X)$ preserving group invertibility in one direction: $T \in \mathcal{G} \Rightarrow \Phi(T) \in \mathcal{G}$. 
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Theorem [M.M-Oudghiri-Souilah, 2017]

Let $X$ be a real or complex infinite-dimensional Banach space, and let $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then $\phi$ preserves $\mathcal{D}_n(X)$ if and only if there exist a scalar $\alpha \neq 0$ and a bounded invertible linear, or conjugate linear, operator $A$ such that either

$$\Phi(T) = \alpha ATA^{-1} \quad \text{or} \quad \Phi(T) = \alpha AT^*A^{-1}.$$
For $T \in \mathcal{L}(X)$, we denote by $\mathcal{P}_n(T)$ the set of all the poles of order $n$ of its resolvent.
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**Corollary**

Let $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be an additive surjective map, and let $n \geq 2$ be an integer. Then $\mathcal{P}_n(\Phi(T)) = \mathcal{P}_n(T)$ for all $T \in \mathcal{L}(X)$ if and only if one of the following assertions holds:

1. There is a bijective continuous mapping $A : X \to X$, either linear or conjugate linear, such that $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(X)$.

2. $X$ is reflexive and there is a bijective continuous mapping $B : X^* \to X$, either linear or conjugate linear, such that $\Phi(T) = BT^*B^{-1}$ for all $T \in \mathcal{L}(X)$.
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