Some recent developments on proximinality in Banach spaces

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Notion of Proximinality

$X$ – Banach space. $S_X = \{ x : \| x \| = 1 \}$, $B_X = \{ x : \| x \| \leq 1 \}$. Let $M \subseteq X$ be a closed set and $x \in X$. Define $d(x, M) = \inf \{ \| x - m \| : m \in M \}$ and $P_M(x) = \{ m \in M : \| x - m \| = d(x, M) \}$. Then

- $M$ is said to be proximinal if $P_M(x) \neq \emptyset$ for all $x \notin M$. 
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- The set valued map $P_M : X \to 2^M$ is called metric projection.
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- In general, \( P_M \) is a closed convex valued map but not necessarily continuous.

- A hyperplane \( Y = \ker(f) \) is proximinal iff \( f \in NA(X) \): 
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P_Y(x) = \{ x - f(x)z : z \in S_X, f(z) = 1 \}.
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  $P_Y(x) = \{x - f(x)z : z \in S_X, f(z) = 1\}$.

- A finite co-dimensional subspace $Y$ is proximinal only if $Y^\perp \subseteq NA(X)$. 
A subspace $M$ is said to be

1. **Strongly proximinal** if given $\varepsilon > 0, x \in X \exists \delta(\varepsilon, x) > 0$ s.t. $P_M(x, \delta) \subseteq P_M(x) + \varepsilon B_X$.

   $P_M(x, \delta) = \{ m \in M : \| x - m \| < d + \delta \}$. 

Remarks

- In general, (2) $\Rightarrow \leftarrow$ Proximinality; (3) $\Rightarrow \leftarrow$ (2); (4) $\Rightarrow \leftarrow$ (1); (4) $\Rightarrow$ (5), (6).
Various strengthenings of Proximinality

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2. **Ball proximinal** if $M$ is a subspace and $P_{B_M}(x) \neq \emptyset \forall x \notin M$.

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In general, $(2) \nRightarrow$ Proximinality; $(3) \nRightarrow (2); (4) \nRightarrow (1); (4) \Rightarrow (5), (6)$. 
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2. **Ball proximal** if $M$ is a subspace and $P_{BM}(x) \neq \emptyset \forall x \notin M$.

3. **Strongly ball proximal** if $B_M$ is strongly ball proximal.

4. **U-proximinal** if $\exists$ a positive function $\varepsilon(\rho), \rho > 0$ with $\varepsilon(\rho) \to 0$ as $\rho \to 0$ and satisfies

   $$(1 + \rho)B_X \cap (M + B_X) \subseteq B_X + \varepsilon(\rho)(B_X \cap M), \rho > 0$$

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5. **Uniformly proximinal** if for $R, \varepsilon > 0 \exists \delta(\varepsilon, R) > 0$ s.t. $\forall x \in X$ and $m \in M$ satisfying $d(x, M) \leq R, \| x - m \| < R + \delta \exists m' \in M$ with $\| m - m' \| < \varepsilon$ and $\| x - m \| \leq R$.

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6. **Uniformly strongly proximinal on $A \subseteq X$** if $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. $P_M(x, \delta) \subseteq P_M(x) + \varepsilon B_X \forall x \in A$.

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In general, (2) $\nRightarrow$ Proximinality; (3) $\nRightarrow$ (2); (4) $\nRightarrow$ (1); (4) $\Rightarrow$ (5),(6).
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5. **Uniform strong proximinality** is introduced in [Dutta, Shunmugaraj, Thota *Uniform Strong Proximinality and Continuity of Metric Projection* J. Convex Anal. 24(4) (2017)]
This talk is based on the following facts

1. **Transitivity of proximinality**: Let $Y \subseteq Z \subseteq X$, $P_1, P_2$ are one of these proximinality properties, then as a subspace $Y \subseteq X$, $Y$ has similar property or not.
This talk is based on the following facts

1. **Transitivity of proximinality**: Let $Y \subseteq Z \subseteq X$, $P_1, P_2$ are one of these proximinality properties, then as a subspace $Y \subseteq X$, $Y$ has similar property or not.

2. **Interplay between proximinality and ball intersection property**: A subspace having ball intersection property has proximinality. Does any other stronger property also follows from ball intersection property?
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1. **Transitivity of proximinality**: Let $Y \subseteq Z \subseteq X$, $P_1, P_2$ are one of these proximinality properties, then as a subspace $Y \subseteq X$, $Y$ has similar property or not.

2. **Interplay between proximinality and ball intersection property**: A subspace having ball intersection property has proximinality. Does any other stronger property also follows from ball intersection property?

3. **Stability of proximinality in various function spaces**: Q. Does $Y \subseteq X \Rightarrow \mathcal{F}(Y) \subseteq \mathcal{F}(X)$?
Transitivity of proximinality

Results from the literature

[The problem by Pollul] Which nonreflexive space $X$ has the following property?

$Y \subseteq Z \subseteq X$ where $\dim(X/Y) < \infty \iff Y \subseteq X$.

Ans. [Indumathi] $c_0$, any incomplete IPS have this property. Any infinite dim $C(K)$, $L_1(\mathbb{R}, \lambda)$ does not have this property.
Transitivity of proximinality

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- [Indumathi] In $L_1(\mathbb{R}, \lambda)$, $\exists$ a subspace $Y$ s.t. for every subspace $Z \subseteq L_1(\mathbb{R}, \lambda)$, $Y \subseteq Z$ is proximinal in $L_1(\mathbb{R}, \lambda)$ while $Y$ is not proximinal in $L_1(\mathbb{R}, \lambda)$. 
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- [Dutta & Narayana] Let $Y \subseteq SPZ \subseteq C(K)$ where $\text{dim}(C(K)/Y) < \infty$ then $Y \subseteq SP C(K)$. 

- [Garkavi] Let $Y \subseteq PZ \subseteq X$ s.t. $\text{dim}(X/Y) < \infty$ then any subspace $Y \subseteq Z$, $Z$ is proximinal in $X$.
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- [The problem by Pollul] Which nonreflexive space $X$ has the following property?
  
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- [Dutta & Narayana] Let $Y \overset{SP}{\subseteq} Z \overset{SP}{\subseteq} C(K)$ where $\dim(C(K)/Y) < \infty$ then $Y \overset{SP}{\subseteq} C(K)$.

- [Garkavi] Let $Y \overset{P}{\subseteq} X$ s.t. $\dim(X/Y) < \infty$ then any subspace $Y \subseteq Z, Z$ is proximinal in $X$. 
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- [The problem by Pollul] Which nonreflexive space $X$ has the following property?
  
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- [Indumathi] In $L_1(\mathbb{R}, \lambda)$, $\exists$ a subspace $Y$ s.t. for every subspace $Z \subseteq L_1(\mathbb{R}, \lambda), Y \subseteq Z$ is proximinal in $L_1(\mathbb{R}, \lambda)$ while $Y$ is not proximinal in $L_1(\mathbb{R}, \lambda)$.

- [Dutta & Narayana] Let $Y \not\subseteq Z \subseteq C(K)$ where $\dim(C(K)/Y) < \infty$ then $Y \subseteq C(K)$.

- [Garkavi] Let $Y \subseteq X$ s.t. $\dim(X/Y) < \infty$ then any subspace $Y \subseteq Z, Z$ is proximinal in $X$.

- [Cheny & Wulbert] Let $Z \subseteq X$ and $Y \not\supseteq Z$ satisfying $Y/Z \subseteq X/Z$ then $Y \not\subseteq X$. 
Transitivity of proximinality

**Theorem**

Let $Z \subseteq^SP Y \subseteq^M\sum_{\text{mand}} X \implies Z \subseteq^SP X$.
Transitivity of proximinality

**Theorem**

Let $Z \subseteq Y \subseteq X \implies Z \subseteq X$.

**Theorem (Jayanarayan, -; JMAA (426)2015)**

Let $Z \subseteq Y \subseteq X$ where $Z$ is finite codimensional in $X$. Then $Z \subseteq X$. 
Transitivity of proximinality

**Theorem**

Let $Z^{SP} \subseteq Y \subseteq X \implies Z^{SP} \subseteq X$.

**Theorem (Jaynarayan, -; JMAA (426)2015)**

Let $Z^{SP} \subseteq Y \subseteq X$ where $Z$ is finite codimensional in $X$. Then $Z^{SP} \subseteq X$.

**Theorem (Jaynarayan, -; JMAA (426)2015)**

A duality between $Y \subseteq X$ & $Y^\perp \subseteq X^{**}$: $Y^{SP} \subseteq X \iff Y^\perp^{SP} \subseteq X^{**}$.
Transitivity of proximinality

Definition

(a) **The norm on X is strongly subdifferentiable** (SSD in short) at $x \in X$ if the one sided limit
$$\lim_{t \to 0^+} \frac{||x + ty|| - ||x||}{t}$$
exists uniformly for $y \in S_X$. In this case we call $x$ is a SSD point of $X$.

(b) We call $x \in X$ is a **quasi polyhedral** (QP in short) point of $X$ if $\exists \delta > 0$ s.t.
$$J_{X^*}(z) \subseteq J_{X^*}(x) \text{ if } ||z - x|| < \delta, ||z|| = ||x||$$
where $J_{X^*}(x) = \{ f \in B_{X^*} : f(x) = 1 \}$. 

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\[\blacksquare\] $x$ is a QP point $\Rightarrow$ $x$ is a SSD point; But the converse is not true.
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**Theorem**

For $f \in S_{X^*}$, suppose $Y = \ker(f)$ then, $Y^{SP} \subseteq X \iff f$ is SSD point of $X^*$. 
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**Theorem (Jayanarayan, -; JMAA (426)2015)**

For a positive measure $\mu$, $f \in L_1(\mu)$ is a SSD point $\iff f$ is a QP point.
Transitivity of proximinality

Proposition (Jayanarayan, -; JMAA (426)2015)

If $X$ is an $L_1$-predual and $Y \subseteq X$ be a finite co-dimensional proximinal subspace of $X$. the following are equivalent:

(a) $Y$ is strongly proximinal.
(b) $Y \perp \subseteq \{x^* \in X^*: x^* \text{ is an SSD-point of } X^*\} = \{x^* \in X^*: x^* \text{ is a QP-point of } X^*\}.$

■ Question: It is clear that, a finite co-dimensional subspace $Y$ of $X$, $Y \subseteq X \implies Y \perp \subseteq \{x^* \in X^*: x^* \text{ is an SSD-point of } X^*\}$
Is the converse also true?

■ Question: Suppose $Z \subseteq Y \subseteq X$ & $\dim(X/Z) < \infty$ then is it necessary that $Z \subseteq X$?
Some useful Definitions

Definition

(a) A subspace $Y$ of a Banach space $X$ is said to have the (strong)$n$-ball property if, whenever $\{B[a_i, r_i]\}_{i=1}^n$ are closed balls in $X$ with $\cap B[a_i, r_i] \neq \emptyset$ and $Y \cap B[a_i, r_i] \neq \emptyset$, then $Y \cap (\cap_i B[a_i, r_i + \varepsilon]) \neq \emptyset$ for every $(\varepsilon = 0)\varepsilon > 0$.

(b) A subspace $Y$ of a Banach space $X$ is said to have the (strong)$1 \frac{1}{2}$-ball property if (a) is true for $n = 2$ and one of the centres of the balls is in $Y$.

(c) A subspace $Y$ is a semi M-ideal in $X$ if there is a (nonlinear) projection $P$ from $X^*$ onto $Y^\perp$ such that for all $x^*, y^* \in X^*, \lambda \in \mathbb{R}$

$$P(\lambda x^* + Py^*) = \lambda Px^* + Py^*$$

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\|.$$
Results from the literature

- [Yost] Y has $n$ ball property $n \in \mathbb{N} \cup \{1\frac{1}{2}\} \Rightarrow Y$ has best approximation property; in fact $Y$ is strongly proximinal.
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- [Yost] $Y$ has $n$ ball property $n \in \mathbb{N} \cup \{1 \frac{1}{2}\} \Rightarrow Y$ has best approximation property; in fact $Y$ is strongly proximinal.

- Let $Y$ be an $M$-summand in a Banach space $X$ and $Z$ be a subspace of $Y$. Let $n \in \mathbb{N} \cup \{1 \frac{1}{2}\}$. If $Z$ has (strong) $n$-ball property in $Y$ then $Z$ has (strong) $n$-ball property in $X$. 
proximinality and ball intersection property

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[Yost] Let $Y \supseteq Z \supseteq X \Rightarrow Y \supseteq \min\{n,m\}$.
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$\frac{1}{2} \text{ ball} \subseteq X \Rightarrow Y \subseteq X$.
proximinality and ball intersection property

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- [Yost] Let $Y \subseteq Z \subseteq X \Rightarrow Y \subseteq \min\{n,m\} \subseteq X$.

- $Y \subseteq X \Rightarrow Y \subseteq X$.

- [Yost] $Y \subseteq X \Rightarrow P_Y$ is continuous, homogeneous, quasi additive map.
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[Yost] Let $Y \subseteq^n Z \subseteq^m X \Rightarrow Y \subseteq^\min\{n,m\} X$.

\[ Y \subseteq^{1\frac{1}{2}} X \Rightarrow Y \subseteq\text{U prox} X. \]

[Yost] $Y \subseteq^{1\frac{1}{2}} X \Rightarrow P_Y$ is continuous, homogeneous, quasi additive map.

[Yost] $1\frac{1}{2}$ ball property $+ \text{Unique Hahn-Banach extn property} \iff 2$ ball property.
Theorem (Jayanarayan, -; JMAA (426)2015)

A duality between $Y \subseteq X$ & $Y^\perp \subseteq X^{**}$: $Y \subseteq X \iff Y^\perp \subseteq X^{**}$.
Proximinality and ball intersection property

**Theorem (Jayanarayan, -; JMAA (426)2015)**

A duality between $Y \subseteq X$ & $Y \perp \perp \subseteq X^{**}$: $Y^{n-ball} \subseteq X \iff Y^{\perp \perp}^{n-ball} \subseteq X^{**}$.

**Theorem (Jayanarayan, -; JMAA (426)2015)**

Let $X$ has 3.2.I.P. and $Y^{3ball} \subseteq X \implies Y^{strong3ball} \subseteq X$. 

Theorem (Jayanarayan, -; JMAA (426)2015)

A duality between $Y \subseteq X$ & $Y^\perp \subseteq X^{**}$: $Y^{n-ball} \subseteq X \iff Y^\perp \subseteq X^{**}$.

Theorem (Jayanarayan, -; JMAA (426)2015)

Let $X$ has 3.2.I.P. and $Y^{3ball} \subseteq X = \Rightarrow Y^{strong3ball} \subseteq X$.

Theorem (-, AOT, no2, 2017)

$Y^{\frac{1}{2} ball} \subseteq X \Rightarrow Y^{unif\ prox} \subseteq X$.
Proximinality and ball intersection property

**Theorem (Jayanarayan, -; JMAA (426)2015)**

A duality between $Y \subseteq X$ & $Y^\perp \subseteq X^{**}$: $Y^n-ball \subseteq X \iff Y^\perp n-ball \subseteq X^{**}$.  

**Theorem (Jayanarayan, -; JMAA (426)2015)**

Let $X$ has 3.2.I.P. and $Y 3-ball \subseteq X \implies Y$ strong$3-ball \subseteq X$.  

**Theorem (-, AOT, no2, 2017)**

$Y^{1\frac{1}{2}}-ball \subseteq X \implies Y$ unif prox $\subseteq X$.  

**Theorem (-, AOT, no2, 2017)**

$X$ has 3.2.I.P + $Y 3-ball \subseteq X \implies B_Y$ unif prox $\subseteq X \implies Y$ unif prox $\subseteq X$.  


Stability of proximinality

Results from the literature

[Jose Mendozá, JAT;1998] Let $(\Omega, \Sigma, \mu)$ be a complete, $\sigma$-finite measure space, $Y$ be a subspace of $X$, $p \in [1, \infty]$ and $f \in L_p(\mu, X)$, then

(a) $d(f, L_p(\mu, Y)) = \|d(f(\cdot), Y)\|_p$.

(b) $f \in P_{L_p(\mu, Y)}(g) \iff f(t) \in P_Y(g(t))$ a.e.

(c) $L_\infty(\mu, Y)$ is proximinal in $L_\infty(\mu, X) \iff$ for $f \in L_\infty(\mu, X)$ there exists $g \in L_\infty(\mu, Y)$ such that $f(t) \in P_Y(g(t))$ a.e.
Stability of proximinality

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[Jose Mendozá, JAT;1998] Let $(\Omega, \Sigma, \mu)$ be a complete, $\sigma$-finite measure space, $Y$ is separable subspace of $X$, then for $1 \leq p \leq \infty$

$L_p(I, Y)$ is proximinal in $L_p(I, X) \iff Y$ is proximinal in $X$. 
Stability of proximinality

**Lemma (-, AOT, no2, 2017)**

Let $Y$ be a strongly proximinal subspace of $X$ and $f \in L_p(I, Y), g \in L_p(I, X)$ then

(a) $d(f, P_{L_p(I,Y)}(g)) = \left( \int_0^1 d(f(t), P_Y(g(t)))^p \, dm(t) \right)^{1/p}$, for $1 \leq p < \infty$.

(b) $d(f, P_{L_\infty(I,Y)}(g)) = \|d(f(.), P_Y(g(.)))\|_\infty$. 


Lemma (−, AOT, no2, 2017)

Let $Y$ be a strongly proximinal subspace of $X$ and $f \in L_p(I, Y), g \in L_p(I, X)$ then

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Theorem (−, AOT, no2, 2017)

$Y^{SP} \subseteq X \iff L_p(I, Y)^{SP} \subseteq L_p(I, X)$, for $1 \leq p < \infty$. 
Stability of proximinality

Lemma (-, AOT, no2, 2017)

Let $Y$ be a strongly proximinal subspace of $X$ and $f \in L_p(I, Y), g \in L_p(I, X)$ then

(a) $d(f, P_{L_p(I,Y)}(g)) = \left( \int_0^1 d(f(t), P_Y(g(t)))^p \, dm(t) \right)^{1/p}$, for $1 \leq p < \infty$.

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Theorem (-, AOT, no2, 2017)

$Y^{SP} \subseteq X \iff L_p(I, Y)^{SP} \subseteq L_p(I, X)$, for $1 \leq p < \infty$.

Theorem (-, AOT, no2, 2017)

$B_Y^{SP} \subseteq X \iff B_{L_p(I,Y)}^{SP} \subseteq L_p(I, X)$, for $1 \leq p < \infty$. 
Stability of proximinality

Theorem (\textendash{}, AOT, no2, 2017)

\[ B_Y^p \subseteq X \iff B_{L_p(I,Y)}^p \subseteq L_p(I,X), \text{ for } 1 \leq p \leq \infty. \]
Stability of proximinality

Theorem (-, AOT, no2, 2017)

\[ B^p_Y \subseteq X \iff B^p_{L_p(I,Y)} \subseteq L_p(I,X), \text{ for } 1 \leq p \leq \infty. \]

Theorem (-, AOT, no2, 2017)

\[ Y \underset{unif\;prox}{\subseteq} X \implies L_{\infty}(I, Y) \underset{SP}{\subseteq} L_{\infty}(I, X). \]
Stability of proximinality

**Theorem (-, AOT, no2, 2017)**

\[ B_Y^P \subseteq X \iff B_{L_p(I, Y)}^P \subseteq L_p(I, X), \text{ for } 1 \leq p \leq \infty. \]

**Theorem (-, AOT, no2, 2017)**

\[ Y \text{ unif prox } \subseteq X \implies L_\infty(I, Y) \text{ SP } \subseteq L_\infty(I, X). \]

**Theorem (-, AOT, no2, 2017)**

Let \( Y \) be a subspace of \( X \) having (strong)\( n \)-ball property, then \( L_p(I, Y) \) has (strong)\( n \)-ball property in \( L_p(I, X) \), \( p = 1, \infty \) and \( n \geq 1 \frac{1}{2} \).
References

THANK YOU!