Solve any four of the following six problems. Please write carefully and give sufficient explanations.

Problem 1

For $1 \leq p \leq \infty$, let $l^p = L^p(\mathbb{N}, \mu)$ where $\mu$ denotes the counting measure. Hence $l^p$ consists of sequences $(x_n)$ such that $\|(x_n)\|_p = (\sum_{k=1}^{\infty} |x_n|^p)^{1/p} < \infty$ in case $1 \leq p < \infty$ and $\|(x_n)\|_\infty = \sup\{|x_n| \mid n \in \mathbb{N}\} < \infty$ in case $p = \infty$.

Let $c_0$ be the closed subspace of $l^1$ consisting of all sequences that converge to 0.

(a) Prove: If $(y_n)$ is a sequence in $l^1$ and $f$ is defined for $(x_n)$ in $c_0$ by $f((x_n)) = \sum_{k=1}^{\infty} x_k y_k$, then $f$ is a bounded linear functional on $c_0$ and $\|f\| = \|(y_n)\|_1$.

(b) Conclude that $c_0^*$, the dual space of $c_0$, and $l^1$ are isometrically isomorphic.

(c) Prove: Every sequence $(y_n)$ in $l^1$ gives rise to a bounded linear functional on $l^1$ as in part (a).

However, there is a non-zero bounded linear functional on $l^1$ that vanishes on all of $c_0$.

Problem 2

Let $\{e_k \mid k \in \mathbb{N}\}$ and $\{f_k \mid k \in \mathbb{N}\}$ be two orthonormal bases of $L^2(0,1)$. Let $X = \text{span}\{e_k \mid k \in \mathbb{N}\}$, endowed with the $L^2$-norm. Hence $X$ consists of finite linear combinations of the basis elements $e_k$.

Consider the linear operator $T : X \rightarrow L^2(0,1)$, defined by $Te_k = k f_k$, $k \in \mathbb{N}$.

Show that $T$ is an unbounded, injective operator, that its inverse is bounded and that the range of $T$ is dense in $L^2(0,1)$.

Problem 3

Let $(f_n)$ be a sequence of real valued, integrable functions defined on a measurable subset of $\mathbb{R}$, denoted by $E$. Prove that, if $\sum_{n=1}^{\infty} \int_E |f_n| < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges almost everywhere on $E$ to an integrable function $f$ and $\int_E f = \sum_{n=1}^{\infty} \int_E f_n$.

Problem 4

Let $[a, b]$ be a closed interval in $\mathbb{R}$. Denote by $BV_0$ the set of all functions of bounded variation on $[a, b]$ that vanish at $a$. For each function $f$ on $[a, b]$, set $\|f\| = V_a^b f$ where $V_a^b$ denotes total variation on $[a, b]$. Prove:

1. The product of two functions in $BV_0$ is in $BV_0$.
2. $\|f\| = V_a^b f$ defines a norm on $BV_0$.
3. A Cauchy sequence of functions in $BV_0$ converges to a function in $BV_0$. 
Problem 5

Prove:
(a) If \( f \) is an integrable function defined on a measurable set \( E \subset \mathbb{R} \), then the set
\[
A = \{ x \in E : f(x) \neq 0 \}
\]
is \( \sigma \)-finite (i.e. it can be written as a countable union of measurable sets, each with finite measure).
(b) Suppose \( f \) and \( g \) are measurable functions on \( \mathbb{R} \). If \( \sqrt{f^2+g^2} \) is integrable then \( fg \) is integrable.

Problem 6

(a) Define what it means for a real function to be absolutely continuous on a closed interval \([a, b] \subset \mathbb{R}\).
(b) Let \( f \) be a continuous function on \([a, b]\) of bounded variation such that \( f \) is absolutely continuous on \([a, c]\) for all \( c \in (a, b) \). Prove that \( f \) is absolutely continuous on \([a, b]\).
(c) Let \( f \) be defined on \([0, 1]\) by \( f(x) = x^2 \sin(1/x^2) \) for \( x \neq 0 \) and \( f(0) = 0 \). Prove that \( f \) has finite derivative at every point. Is \( f \) absolutely continuous? Explain your answer.