Master Exam-2005 Fall
Algebra I

Answer any three of the following six problems.

1. Let M, N be two normal subgroup of G such that $M \cap N = \{e\}$. Show that for any $m \in M$ and $n \in N$, $mn = nm$.

2. Let H be a normal p-subgroup of a (finite) group G. Prove that H is contained in every Sylow p-subgroup of G.

3. Let H be a subgroup of G and let $k = \frac{|G|}{|H|}$ be the index of H in G. Suppose that $k! < |G|$. Show that H contains a nontrivial normal subgroup of G.

4. Let R be a commutative ring with identity. Show that M is a maximal ideal of R if and only if the quotient ring R/M is a field.

5. State the definition of Euclidean Domain and prove that every Euclidean domain is a Principal Ideal Domain.

6. Let R be the ring of all continuous functions on $[0, 1]$ and let I be the collection of functions $f(x)$ in R with $f(1/2) = f(2/3) = 0$. Prove that I is an ideal of R but is not a prime ideal.