Masters Exam (Spring 2003)
Algebra II

Answer any **three** of the following six questions.
You should state clearly any general results you use.

1. What is the Galois group of $X^3 - 35X + 15$ over

   (a) $\mathbb{F}_2$ (the field of 2 elements),

   (b) $\mathbb{F}_3$ (the field of 3 elements),

   (a) $\mathbb{Q}$.

   State clearly any results you use.

2. (a) State the tower law for finite field extensions.

   (b) Let F/K be a finite extension and let $\alpha, \beta \in K$. If $[F(\alpha) : F]$ and $[F(\beta) : F]$ are relatively prime, show that $[F(\alpha, \beta) : F] = [F(\alpha) : F][F(\beta) : F]$

   (c) Give an example that shows that 'relatively prime' cannot be omitted in (b)

3. Let K be the splitting field of $f(X) = X^4 - 3X^2 - 10$ over $\mathbb{Q}$. Determine all the subfields of K, the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, and give the Galois correspondence between the subfields of K and the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$.

4. Let K/F be a Galois extension with $[K : F] = 4$. Show that there is a field L with $F \subseteq L \subseteq K$ and $[L : F] = 2$.

5. Suppose A is a finitely generated abelian group and $A \oplus A \cong A$. Show that $A = 0$. Give an example of an abelian group $A \neq 0$ with $A \oplus A \cong A$.

6. Show that every module is the quotient of some free module.