Master Exam Part II (Real Analysis) -Summer 2010

Do three of following problems.

1. Let $f(x)$ be an integrable function on $\mathbb{R}$ and for each n, let $h_n(t) = \frac{t^{2n}}{2+t^{2n}}$. Show that the limit $\int_{n\to\infty} f(t)h_n(t)dt$ exists and find its limit.

2. The outer measure on $\mathbb{R}$ is defined by

$$m^*(A) = \inf\{\sum_{n=1}^{\infty} \ell(I_n) : \ I_n \text{ are open intervals such that } A \subseteq \cup_n I_n\}.$$

   ($\ell(I)$ denotes the length of the interval I) Show that for any countable subsets E_n of $\mathbb{R}$, $m^*(\cup_n E_n) \leq \sum_n m^*(E_n)$. (countable subadditivity).

3. Let $\{f_n\}$ be a sequence of measurable function that converges to f in measure. Show there is a subsequence of $\{f_n\}$ that converges to f a.e.

4. State the Minkowski and Hölder inequalities.

5. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ such that f_n converges to f a.e. Suppose that f is integrable and $\lim_{n\to\infty} \int f_n(t)dt = \int f(t)dt$. Show that for any measurable subset E of $\mathbb{R}$,

$$\lim_{n\to\infty} \int_E f_n(t)dt = \int_E f(t)dt.$$