Masters Comprehensive exam
Complex Analysis

February 2013

Answer **three** of the following five questions.

1. Prove or disprove the existence of an analytic function f defined on $\{z : |z| < 2\}$ such that
$$f\big(\tfrac{1}{n}\big) = f\big(-\tfrac{1}{n}\big) = \tfrac{1}{n^3}, \qquad n = 1, 2, 3, \ldots$$

2. (a) State Rouché's Theorem.
   (b) Prove or disprove the existence of a sequence of analytic functions f_n defined on $\{z : |z| < 2\}$ such than $f_n(z) \to \overline{z}$ uniformly on the circle $\{z : |z| = 1\}$?

3. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1}\, dx.$$

4. Find a meromorphic function f with simple poles at $z = \sqrt{n}$, $n = 1, 2, 3, \ldots$, with residues $\mathrm{res}_{z=\sqrt{n}}(f) = 1$.

5. (a) State the Maximum Modulus Principle.
   (b) Suppose $|z_i| = 1$ for $i = 1, 2, \ldots, n$. Prove that $\prod_{i=1}^{n} |z - z_i| > 1$ for some z with $|z| = 1$.