Topology

April 6, 2002
Master's Exam

Solve only three out of the following five problems.

1. Prove that a compact subset of a Hausdorff topological space is closed.

2. Let R^N represent the cartesian product of a countable and infinite number of copies of R with the usual topology. Let Γ be the product topology on R^N. Answer only one of the following two questions:

   (a) Is $A = \{\{x_i\}_i,\, x_1 \times x_2 > 0\}$ open? Explain your answer.

   (b) Is the set of all eventually constant sequences closed? Explain your answer.

3. Is a circle homeomorphic to the figure "eight"(both sets with the usual topology induced from the plane)? Explain your reasoning.

4. Let M be a connected topological manifold and x a point in M.

   (a) Define the fundamental group of M based at x, $\Pi_1(M,\, x)$.

   (b) Show that $\Pi_1(M,\, x)$ is independent of the point x up to group isomorphisms. [Hint: Start by showing that M is path connected.]

5. Let $(X,\, d)$ be a compact metric space and $f : X \to X$ is a contraction (i.e. there exists $0 < \alpha < 1$ such that for every x and y in X $d(f(x),\, f(y)) < \alpha\, d(x,\, y)$) then there exists a unique $x \in X$ such that $f(x) = x$.