Masters' Comprehensive Exam: Topology

September 20, 2008

Do any **three** *of the following four problems. You have* **1 hour**.

1. (a) Prove or disprove that the subspace topology on the set of integers, $\mathbb{Z}$, in $\mathbb{R}$ (with its usual topology) is the same as the discrete topology on $\mathbb{Z}$.

   (b) Prove or disprove that the subspace topology on the set of rationals, $\mathbb{Q}$, in $\mathbb{R}$ (with its usual topology) is the same as the discrete topology on $\mathbb{Q}$.

2. Let X be a metric space. Show that the following are equivalent:

   (a) X has a countable basis;

   (b) X is Lindelöf (i.e., every open cover of X has a countable subcover);

   (c) X has a countable dense subset.

3. Suppose that $A \times B \subseteq U \subseteq H \times K$, where A, B, H, and K are compact Hausdorff spaces and U is open in the product space $H \times K$. Show that there exist open subsets V of H and W of K such that $A \times B \subseteq V \times W \subseteq U$.

4. Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on a closed interval and let $G = \{(x, f(x)) \mid x \in [a,b]\} \subseteq \mathbb{R}^2$ be its graph. Prove or give a counterexample for the following.

   (a) If G is connected, then f is continuous.

   (b) If f is continuous, then G is connected.