Master Exam Part II-Topology (Spring 2010)

Do three of the following four problems.

1. Let (X, d) be a metric space. Show that every closed ball is closed, i.e., for any $x \in X$ and $a > 0$, the set
$$\{y : d(x, y) \leq a\}$$
is closed.

2. Let f be a continuous function from X to Y and let C be a connected subset of X. Show that $f(C)$ is a connected subset of Y.

3. Let X be a compact Hausdorff topological space. Show that X is regular, i.e., for any nonempty closed subset C of X and a point $x \notin C$, there are two disjoint open subsets O, U of X such that $x \in O$ and $C \subseteq U$.

4. Let X be a metric space. Show that if X is separable, then X is second countable.