MASTER'S EXAM (SUMMER 2002)
BASIC EXAM

Answer any **six** of the following eight questions.

1. (a) State and prove the intermediate value theorem.
   (b) Consider a continuous function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = f(1)$. Using the intermediate value theorem or otherwise, prove that there exists $x \in [0, \frac{1}{2}]$ such that $f(x) = f(x + \frac{1}{2})$.

2. Let $f(x)$ be a continuous function from $\mathbb{R}$ to itself such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (a) Prove that $f(\frac{p}{q}) = \frac{p}{q} f(1)$ for all rational numbers $\frac{p}{q}$.
   (b) Show how to deduce that $f(x) = x f(1)$ for all real numbers x.

3. Let X be a compact metric space and $f\colon X \to \mathbb{R}$ be a continuous function.
   (a) Prove that f is bounded above (i.e., there exists an M such that $f(x) \leq M$ everywhere).
   (b) Prove that f attains its bounds (i.e., there exists $x_0 \in X$ such that $f(x) \leq f(x_0)$ everywhere).

4. (a) What does it mean to say that a topological space is Hausdorff?
   (b) Show that any metric space is Hausdorff.
   (c) List all topologies on the set $\{a, b\}$.
   (d) Of the topologies listed in part (c), which ones are Hausdorff?

5. Let V be the collection of real polynomials of degree less than or equal to 2.
   (a) Show that with the usual operations of addition of polynomials and multiplication by real numbers, the set V forms a vector space over the reals.
   (b) Consider the map $T\colon V \to \mathbb{R}^3$ defined by $T(p) = (p(0), p(1), p(2))$. Show that T is surjective.
   (c) Is T injective?

Please turn over

6. (a) Prove that if U and W are finite-dimensional subspaces of a vector space V then

$$\max(\dim(U), \dim(W)) \leq \dim(U + W) \leq \dim U + \dim(W)$$

[Note: If you use any formula for $\dim(U + W)$ you should prove it].

(b) Give an example of a pair of (non-zero) subspaces U and W such that

$$\max(\dim(U), \dim(W)) = \dim(U + W).$$

(c) Give an example of a pair of (non-zero) subspaces U and W such that

$$\dim(U + W) = \dim U + \dim(W).$$

7. State and prove the theorem of Lagrange concerning subgroups of finite groups.

8. Let G be a group. Let H be the subgroup of G generated by elements of G of the form $xyx^{-1}y^{-1}$ for x, y belonging to G.

(a) Prove that H is a normal subgroup of G.

(b) Prove that G/H is Abelian.