Master's Exam: Core Topics

Spring 2003

Answer any Six Questions; credit will be given for the best six questions
You must state clearly any general results you use. Explain your answers and show all working

1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. For each of the following statements, prove it or give a counterexample:
   (a) If $\sum_{n=1}^{\infty} a_n$ exists then $\lim_{n\to\infty} a_n = 0$.
   (b) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ exists.
   (c) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ exist then $\sum_{n=1}^{\infty}(a_n + b_n)$ exists.

2. Prove **direct from the definition** that the function $f(x) = 1/(x+1)$ is continuous at each point $a \in \mathbb{R} \setminus \{-1\}$.

3. Consider the real-valued function defined as follows:
$$\begin{array}{rcl} f:\,(0,\,\infty) & \to & R \\ x & \to & \frac{1}{x} \end{array}.$$
   (a) Show that f is not uniformly continuous.
   (b) Characterize those subsets $A \subseteq (0,\,\infty)$ such that f restricted to A is uniformly continuous.

4. Consider the reals with the usual topology and $A = (0,\,1] \cup \{1 + \frac{1}{n}\}_{n=1,\,2,\,\ldots}$ with the induced topology.
   (a) Is the interval $(0,\,1]$ a closed subset of A?
   (b) Determine the interior and the closure, in A, of $\{1 + \frac{1}{n}\}_{n=1,\,2,\,\ldots}$.

5. (a) Let G be a finite group and let H be a subgroup of G of index 2. Show that H is a normal subgroup of G.

   (b) Give an example of a group G and a subgroup H of index 3 that is not normal.

6. Let G be a group and let H be the subgroup generated by all expressions of the form $xyx^{-1}y^{-1}$.

   (a) Show that H is a normal subgroup of G.

   (b) Show that the quotient G/H is abelian.

7. Let U and V be two real vector spaces and let $\mathrm{Hom}(U, V)$ be the set of linear maps from U to V.

   (a) Show that $\mathrm{Hom}(U, V)$ is a real vector space under pointwise addition and scalar multiplication, $(f + g)(x) = f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$.

   (b) Show that $\dim \mathrm{Hom}(U, V) = (\dim U)(\dim V)$.

8. Let $T \colon V \to V$ be a linear map on a real vector space V such that $T^2 = 1$.

   (a) Show that $V_+ = \{x : Tx = x\}$ and $V_- = \{x : Tx = -x\}$ are subspaces of V.

   (b) Show that V is isomorphic to the direct sum of V_+ and V_-.