Master's Exam: Basic Areas

March 13th 2004

Answer any **five** *questions.*
Show all working; State clearly all theorems that you apply.

1. Define a sequence (x_n) by $x_0 = 1$, $x_{n+1} = (x_n/2) + \sqrt{x_n}$. Prove that the sequence (x_n) is convergent and identify the limit.

2. (a) State and prove the comparison test for convergence of series.

   (b) Prove that the series
$$\sum_{n=2}^{\infty} \frac{n + 5\sqrt{n}}{n^{5/2} - 1}$$
   is convergent.

3. Define for $x > 0$, $L(x) = \int_1^x \frac{1}{t}\, dt$. Without assuming any properties of logarithms, show that

   (a) $L(x)$ is a strictly increasing function, i.e., $x_1 < x_2$ implies $L(x_1) < L(x_2)$.

   (b) $L(xy) = L(x) + L(y)$ for $x, y > 0$.

   (c) $L(x)$ tends to ∞ as $x \to \infty$.

4. (a) Define what it means for a subset of a metric space to be compact.

   (b) Show that the image of a continuous function on a compact set is compact.

   (c) If C is a compact subset of a metric space X, show that for any point $x \in X$ there is a point $y \in C$ such that the distance $d(x, y)$ is minimal, i.e., $d(x, y) \leq d(x, z)$ for all $z \in C$.

Turn over

5. Let U_i be subspaces of a finite dimensional vector space V.
   (a) Show that $\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2)$.
   (b) Give an example that shows that $\dim(U_1 + U_2 + U_3) + \dim(U_1 \cap U_2 \cap U_3)$ need not be equal to $\dim(U_1) + \dim(U_2) + \dim(U_3)$ in general.

6. The normalizer N_H of a subgroup H of a group G is defined as $\{g \in G : gH = Hg\}$.
   (a) Prove that H is a normal subgroup of N_H.
   (b) By considering a suitable subgroup of the dihedral group with 6 elements, show that N_H need not be a normal subgroup of G.

7. (a) Suppose G is a group in which $x^5 = 1$ for all $x \in G$. Show that either $G = \{1\}$ or the order of G is divisible by 5.
   (b) Give an example of a group $G \neq \{1\}$ in which $x^6 = 1$ for all $x \in G$, but the order of G is not divisible by 6.